Lessons for conformal field theories from bootstrap and holography

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Declaration of Authorship

I, Kallol Sen, declare that this thesis titled, 'Lessons for conformal field theories from bootstrap and holography' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Date: 
The work done in this thesis includes an exploration of both the conformal field theory techniques and holographic techniques of the Gauge/Gravity duality. From the field theory, we have analyzed the analytical aspects of the Conformal Bootstrap program to gain handle on at least a part of the CFT spectrum. The program applies equally to the strongly coupled as well as the weakly coupled theories. We have considered both the regimes of interest in this thesis. In the strongly coupled sector, as we have shown that it is possible to extract information about the anomalous dimensions, of a particular subset of large spin operators in the spectrum, as a function of the spin and twist of these operators. The holographic analog of the anomalous dimensions from CFT are the binding energies of generalized free fields in the bulk, which has also been analyzed in this thesis. On the contrary, in the weakly coupled sector, the same idea can be used to calculate the anomalous dimensions of operators, with any spin and dimension in an $\epsilon$--expansion. We have considered a simple set of scalar operators, whose anomalous dimensions are reproduced correctly up to $O(\epsilon^2)$. In another holographic calculation, we have analyzed generic higher derivative theories of gravity, which corresponds to boundary theories with infinite colors but finite 't Hooft coupling. Certain universal aspects of these theories, such as anomalies and correlation functions are also calculated. The three point functions for these higher derivative theories will serve as a building block for considering four point functions for finitely coupled boundary CFTs. In the conclusion, we have pointed out the directions of interest which could be locating the bulk duals of large $N$ finitely coupled theories, or that of an intermediate theory with both finite 't Hooft coupling as well as finite gauge group, with a speculative string theory dual.
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For my parents and sister...
Chapter 1

Introduction

In general, symmetries of any system constrain the dynamics of the theory severely. For example, Lorentz invariance constrains the form of the operators entering the action of the Standard Model in particle physics.

In reality only few symmetries are exact and most of the time, the breaking of symmetries are associated with some small parameter which depends on the system under consideration and the resolution we are interested in. Consider a simple example of scale invariance. If the regime of resolution is much larger that the inter-atomic distance, then the microscopics of the theory are irrelevant and the results at this resolution is independent of the system specification – measurements are approximately scale invariant. Even if the parameters associated with the symmetry breaking are dimensionful, the system is approximately scale invariant at length scales much higher or lower than the characteristic length scale of the system and the effect of such parameters are negligible.

Under certain conditions, even in real systems, apart from Lorentz symmetry, there is also an approximate scaling symmetry. Examples of such systems in condensed matter are the 3d Ising model at the critical point. In high energy physics, this kind of scale invariance is exhibited in the low energy effective theory of Quantum Chromodynamics where the heavy quarks decouple from the theory.

Primarily, scale invariance leads to conformal invariance in most of the cases. Although an exact statement of the connection of the two is still lacking except for special case of unitary two dimensional conformal field theories and also four dimensional field theories [1].

In light of the above discussion, the approximate conformal invariance (implied from scale invariance) becomes relevant in the following regimes:

1. In the high energy regime (UV), where the average kinetic energy is much larger than any characteristic mass scale in the theory.
2. In the low energy regime (IR), where the massive particles are heavy and decouple from the theory.

3. For the massless theories, where there is no characteristic mass scale in the theory.

In Renormalization Group flow of Quantum Field theories, where a theory flows from the UV to IR regime, the conformal invariance shows up at certain points in the space of couplings during the flow. These are called fixed points, defined as the set of special points in the space of couplings where the $\beta(\lambda)$ function vanishes ($\lambda$ is the coupling). The possible macroscopic behavior of the system at large scale is defined by its fixed points. The conformal behavior of the fixed points can be shown to all orders in the perturbation by considering the Ward identities of the renormalized stress tensor (see [2]). Hence at the fixed points of the theory, one can use the techniques of conformal field theories to gather knowledge about the general macroscopic features of the theory that do not depend on the knowledge of microstates. This proves to be useful for most strongly coupled theories where perturbative techniques are not available.

As an example, consider the action of a massless scalar theory with $\phi^4$ interaction in four dimensions,

$$S = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{4!} \phi^4 \right]. \quad (1.1)$$

In the limit $\lambda \to 0$, the system describes a free massless scalar field, where the beta function $\beta(0) = 0$. This is called the Gaussian or the UV fixed point. In presence of a non zero coupling, the theory flows into an interacting fixed point, where again $\beta(\lambda) = 0$ for non zero $\lambda$. This is called the interacting or the Wilson-Fisher fixed point. At the interacting fixed point, one can use the CFT techniques, to gain insights into the strongly coupled regime and infer the universal properties of the theory.

In general, conformal invariance is strong enough to restrict the form of the basic two point and three point correlation functions in the theory. But even then, conformal invariance is not enough to fully determine the spectrum of the theory. As we will point out in chapter 2, the conformal field theory generators,

$$P_\mu, \ M_{\mu\nu}, \ D \text{ and } K_\mu, \quad (1.2)$$

define a $SO(d,2)$ conformal algebra. Further, as we will demonstrate later, the action of the generators on correlation functions is constraining enough for the two point functions and the three point functions. For example, for identical scalar operators in CFT, we have,

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \text{const.} \frac{1}{|x - y|^{2\Delta_\phi}}, \quad \langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z) \rangle = \frac{C}{|x - y|^{\Delta_\phi} |y - z|^{\Delta_\phi} |x - z|^{\Delta_\phi}}, \quad (1.3)$$

where $\Delta_\phi$ is the conformal dimension of the scalar operator. But to gain a complete knowledge of the spectrum of operators, one needs to consider higher point correlation functions of the theory since the information about the specific operators and their couplings appear as an information
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3

in these higher point functions. The additional information about the theory comes from the familiar Operator Product Expansion, which involves the operators appearing in the spectrum along with specific couplings (the OPE coefficients). One advantage of the OPE in conformal field theory is that, unlike in ordinary QFT where the OPE does not converge, the OPE of the CFT is a convergent power series for any finite separation, the radius of convergence being an open disc with no other operator insertion. The generic form of the OPE is given by,

\[ \mathcal{O}_i(x)\mathcal{O}_j(y) \in \sum_{k,\Delta,\ell} C_{ij,\Delta,\ell}^k(x-y, \partial_y)\mathcal{O}_k(y). \] (1.4)

where in principle an infinite set of operators \( \mathcal{O}_k \) are allowed in the OPE. The OPE contains information about the operator content in the theory as well as the couplings appearing in the function \( C_{ij,\Delta,\ell}^k(x-y, \partial_y) \) which is fixed by conformal invariance up to some overall constants. The next simplest higher point function is the four point correlator which for generic operators \( \mathcal{O}_i \) is,

\[ \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4) \rangle. \] (1.5)

Using the OPE, the euclidean four point function can be broken into product of the three point functions and squares of the OPE coefficients. The final constraint is the OPE associativity which can be used to constrain the form of the OPE coefficients and the possible operator content of the theory. This constraint is known as the crossing symmetry. The program of determining the CFT spectrum and OPE coefficients this way, is called the Conformal Bootstrap Program. The major part of the thesis will concern elaborating the aspects of the program.

The program of conformal bootstrap is not a new concept, although the modern conventional bootstrap methods to analyze the constraints is a fairly new approach. The basic concepts of the program however dates back to the seminal work of Polyakov [3] in 1973, where he explicitly demonstrated that by using a unitary and crossing symmetric four point correlator and imposing the completeness condition in accordance with the OPE of the theory, one can in principle generate a set of algebraic constraints that can be consistently solved for the dimensions of various operators appearing in the theory and their OPE coefficients. This work was however pertaining to the perturbative calculation of the \( O(n) \) models in \( 4 - \epsilon \) dimensions with an aim to evade the complicated approach of the diagrammatic perturbation theory. However all the ingredients of the modern bootstrap program such as crossing symmetry, unitarity bounds, OPE and so on were inbuilt into the approach.

This concept was resurrected with a new motivation and agenda (specially for \( d > 2 \) dimensions\(^1\)), more than forty years after Polyakov’s work, in several works like [5],[6],[7] using the seminal work on conformal blocks by Dolan and Osborn [8] and since then there has been a renewed interest in various other works which followed in recent years. Using numerical methods, interesting constraints have been placed on conformal field theories in diverse dimensions [5]. Applications have been found in diverse field theories ranging from supersymmetric conformal

\(^1\)For \( d = 2 \), see [4]
field theories [9] to the 3d-Ising model at criticality [10]. The lessons learnt using these methods transcend any underlying Lagrangian formulation and are hoped to be very general. On the other hand, recently there has been advancements in the analytical aspect of the program as well [11, 12, 13] which describes at least a particular part of the CFT spectrum. This thesis will describe the essential ideas of the analytical aspects and provide useful extensions on the analytical side, which sheds light on certain universal properties of the spectrum both for theories at strong and weak coupling.

While bootstrap is one way of handling the strongly coupled regime of QFT, another way of dealing with the strongly coupled regime is the holographic principle which says in some restricted sense that,

“A strongly coupled QFT with the coupling tending to infinity and with a large number of colors, living in d dimensions, can be equivalently described by a (semi) classical theory of gravity living in one higher dimension.”

More specifically, the correspondence is between a strongly coupled \( CFT_d \) with infinite colors and a (semi) classical theory of gravity living in \( AdS_{d+1} \) bulk spacetime. Every field in the bulk space time can be mapped into an operator belonging to the corresponding boundary theory. For example,

\[
\begin{align*}
\text{scalar } \phi & \rightarrow |O\rangle, \text{ scalar operator} \\
\text{gauge field } A_\mu & \rightarrow J_\mu, \text{ current operator} \\
\text{metric } g_{\mu\nu} & \rightarrow T_{\mu\nu}, \text{ stress tensor},
\end{align*}
\] (1.6)

and so on. We should mention here that the most well studied example of the duality known till date is the correspondence between the \( N = 4 \) SU(N) SYM with infinite N and infinite ’t Hooft coupling (\( \lambda \)) living in four dimensional space time and type II B supergravity on \( AdS_5 \times S^5 \). We will provide a brief introduction to the AdS/CFT duality later in the thesis since a part of the thesis will entail calculations from the gravity side as well. Various aspects of the duality such as, the parameters of the two theories of the duality, the spectrum of operators and so on are also discussed in the later chapters. But we have kept the canonical example in mind during the discussion of these aspects. However, we expect these general characteristics of the duality to hold for any boundary theory with infinite coupling and large number of colors, and its corresponding gravity dual. An important part of the thesis is concerned with an alternative of the cumbersome holographic renormalization technique, to determine the one point functions of the stress tensor, using the first law of entanglement (discussed in [14] and also chapter 7) for generic higher derivative theories in the bulk. The implications of the first law of entanglement and universal structures of the one point functions of the stress tensor and the higher point correlation functions therein (up to three point functions), are also discussed in this thesis. These three point functions can serve as building blocks for other types of four point functions involving stress tensors.
As the connection, between the bootstrap program and the holographic work, both of which are a part of the thesis, and which has been emphasized repeatedly in chapter 7 and in the concluding remarks, we emphasize on the onset that the holographic work was the first step towards obtaining the three point correlations of the stress tensor (graviton) from the bulk side, for a generic higher derivative theory of gravity, which corresponds to a finitely coupled boundary theory (a motivation drawn from the canonical example of the duality). Although we do not discuss the structure of the four point correlator of stress tensors from either the bulk or the boundary point of view, anywhere in the thesis, nevertheless the holographic work can be looked as the first step towards the understanding of boundary theories with infinite colors but with finite coupling, thus going beyond the existing holographic dictionary. Further, these three point functions can be used to build the four point functions involving stress tensors and we can ask questions similar to the case of external scalar operators.

1.1 Structure of the thesis

The goal of the study in this thesis is the implementation of the basic ideas of unitarity and crossing symmetry in Conformal Field Theories to restrict the spectrum of the operators appearing in the CFT. The thesis also entails the holographic calculations both from the bootstrap point of view as well as correlation functions for higher derivative bulk theories (corresponding to a large $N$ and finitely coupled boundary QFT) in a general setting.

Thus in chapter 2, we start with a brief introduction to the fundamental aspects of the CFT. We start with the conformal transformations and the conformal algebra therein. We will then discuss the effect of the conformal transformations on the states in the Hilbert space with an elaboration on the primary and the descendant states. We will also consider the effects of finite and infinitesimal conformal transformations on the correlation functions. Using these constraints, it can be shown that the two and the three point correlation functions are completely fixed up to some overall constants. From the four point function onwards, conformal invariance is inadequate and we will need the information about the operator content and their couplings in the spectrum. This information is contained in the Operator Product Expansion. We proceed to discuss this next. Finally we move into the formulation of the conformal blocks using the Casimir differential operators. This gives a differential equation for the conformal blocks for a particular exchange $\mathcal{O}_{\Delta,\ell}$. One final ingredient we need, is the unitarity bound, which we discuss in the next section before moving on to the bootstrap equation. The concluding section in this chapter, discusses the old school bootstrap program started off by Polyakov for determining critical exponents.

In chapter 3, we introduce the AdS/CFT duality. As emphasized in the introduction, this can also be viewed as one plausible way to tackle the strongly coupled CFTs. We start with the
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basics of the AdS space, with an emphasis that the symmetry structure of the AdS isometries follow the same algebra as the $SO(d,2)$ group of the boundary CFT. Then we discuss very briefly, the large $N$ aspects of the CFT and finally, the necessary ingredients of the duality, *i.e.* the parameters, spectrum and correlator matching from of both sides, in some detail.

In chapter 4, we consider the conformal bootstrap method for four identical scalar fields and focus on the large spin sector of the spectrum in four dimensions. We find that in a particular regime of the conformal cross ratios (called the light cone regime), the unity on one side of the bootstrap equation, maps to the large spin double trace operators on the other side. By assuming the existence of the stress tensor in the spectrum, we find that the bootstrap equation is mutually satisfied by an infinite set of large spin operators in the spectrum, with a particular pattern of twists, which includes an additional piece of anomalous dimension. The anomalous dimensions of these large spin operators, depend on the spin of the operator and is an exact polynomial function of the twists and the dimension of the external scalars. For large twists, we find that there is an universal contribution of the anomalous dimension which depends only on the space time dimension and the central charge of the theory.

Chapter 5 is an extension of the the work in chapter 4 to general dimensions. An apparent obstruction to this work was the unavailability of the closed form expressions for the conformal blocks in odd dimensions. But from the recursion relations (at least in the light-cone regime), we were able to extract a closed form expression for the seed conformal block in three dimensions and the conformal blocks in higher dimensions. Using these closed form expressions for the general dimensional conformal blocks, we were able to grind the anomalous dimension for the large spin sector of the spectrum in general dimensions. Again, in the limit of large twists, the anomalous dimensions for these operators followed a universal behavior which depends on the space time dimension and the central charge of the theory. As in the previous work, we also found agreement of the results of the CFT calculation with the bulk, by considering binding energies of free fields in AdS, due to a graviton exchange.

Chapter 6 is a little detour from the conventional bootstrap approach. We instead followed the old bootstrap program à la Polyakov, in which it was shown that using a crossing symmetric and unitary four point correlation function and imposing completeness condition in accordance with the OPE, it is possible to obtain a set of algebraic constraints for the critical exponents of the critical $O(N)$ theory in $4 - \epsilon$ dimensions. We verified the logic in the work and also extended it for a non trivial matching of the anomalous dimensions of the isospin$-0$ and isospin$-2$ scalars upto order $O(\epsilon^2)$.

The final chapter 7 deals with another holographic work in the thesis. A part of the holographic calculation in the thesis, has already entered chapters 4 and 5. But those were in the context
of bootstrap of external scalar operators where we argued, that the holographic analog of the anomalous dimensions of large spin operators in CFT are the binding energies due to interactions of generalized free fields in the bulk. Though a calculation of the other forms of four point correlators like those involving the stress tensors is not presented in this thesis, chapter 7 will present a first step towards this goal. We will compute the one, two and three point functions of the stress tensors (gravitons) for a generic higher derivative gravity theory in the bulk, that describes a CFT with large number of colours ($N$) and with finite ‘t Hooft coupling ($\lambda$). The one point function of the holographic stress tensor is computed using the first law of entanglement, which bypasses the cumbersome holographic renormalization technique. The one point function of the holographic stress tensor becomes proportional to the linear combination of the higher derivative couplings, which in turn can be related to the central charge (related to B-type trace anomalies) of the boundary theory. Using the background field subtraction method, we further calculated the two point functions, and the three point functions of the holographic stress tensor in the shockwave background. The results for the three point functions are in agreement with the forms of the three point correlators of stress tensors in the CFT. Thus, one can use these results as building blocks for any higher (example, four) point functions for stress tensors. Further we can also calculate other four point correlators involving stress tensors and spin-1 conserved currents or scalars in the shockwave background.

We conclude the thesis with some open directions and emphasis on the current topics. The appendices A, B, C and D contain the necessary details of the calculations pertaining to chapters 4, 5, 6 and 7.
Bibliography


Chapter 2

Conformal Field Theory

2.1 Introduction

As described in the introductory chapter regarding the importance of the conformal field theory techniques, in diverse real time areas of physics, both for condensed matter and high energy physics, we will take some time to briefly introduce the basic ideas and the techniques of the conformal field theory. In this chapter, we will mainly follow the presentation in [1, 2] and [3]. However a concise introduction to the topic can also be found in the other references [4, 5, 6] and [7] and many more. The last three references mainly deal with the two dimensional conformal field theories. Since a part of the thesis will involve the program of Conformal bootstrap, we will also introduce this topic in this chapter. Handpicking from the vast amount of literature on this topic is difficult but one can consult [9] for the introductory concepts of the conformal blocks regarding their constructions and so on. Further, regarding the numerical studies on this topic, see [10] while an analytical aspect of the same is discussed in [11].

As emphasized in the previous introduction, under certain circumstances, Quantum field theories enjoy an enhanced symmetry of scale invariance. In most cases (with few exceptions), the scale invariance in equivalent to conformal invariance. When this happens, the symmetry class of the field theory, can be used to infer useful universal properties of that system without the aid of microscopic specifications. Conformal symmetries severely constrain the dynamics of the system under consideration and hence provide an useful tool to extract information in a regime (e.g. strong coupling) where perturbative QFT is no longer valid.

We will use the properties of the stress tensor \((T_{\mu\nu})\) to fully analyze the effect of the conformal symmetry. In general every QFT enjoys a conserved stress tensor defined through,

\[
T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}, \quad \nabla_\mu T^{\mu\nu} = 0. \tag{2.1}
\]
For any diffeomorphism invariant action, the variation with respect to the metric \((g_{\mu \nu})\) can be written as,

\[
\delta S = \int \frac{\delta \mathcal{L}}{\delta g_{\mu \nu}} \delta g_{\mu \nu} = \int T^{\mu \nu} \delta g_{\mu \nu} = 0 .
\]  

(2.2)

With the conformal symmetry, the infinitesimal variation of the metric becomes,

\[
\delta g_{\mu \nu} = \nabla_{\mu} \epsilon_{\nu} + \nabla_{\nu} \epsilon_{\mu} = \Omega(x) g_{\mu \nu} ,
\]  

(2.3)

which implies that,

\[
\delta S = \int \Omega(x) T^{\mu}_{\mu} = 0 .
\]  

(2.4)

For any general \(\Omega(x)\), this implies that,

\[
T^{\mu}_{\mu} = 0 ,
\]  

(2.5)

\textit{i.e.} the stress tensor is traceless when conformal symmetry is present. Thus all conformal field theories have conserved and traceless stress tensor. An additional property for the conformal field theories is the presence of a large number of conserved charges for these symmetries. For example, the dilatation operator implies a conserved Hamiltonian in the radial quantization and so on.

Perhaps the best manifestation of the conformal symmetry is in the correlation functions. It constrains the form of the two and the three point functions up to some overall constant albeit proves inadequate for higher point functions where additional input is required.

All these ideas will be the topic of discussion in the present chapter. We will start with a brief introduction to the conformal symmetry operations and the concerning algebra of the generators. The effect of these transformations on the operators and the correlation functions obtained therein will also be discussed with a brief discussion on the classification of primary operators and their descendants. Then we will move on to the construction of the Hilbert space of the underlying theory. As expected, the Hilbert space is also constrained by conformal symmetry with a particular form of the operator product expansions for the states. One obvious difference with the massive theories is that the OPE for two operators, for scale (conformally) invariant theories include an infinite tower of operators in the spectrum and is convergent for any finite separation of the seed operators (rather than just in the limiting case of vanishing separation) as will be discussed later in this chapter. Finally, we will discuss the foundations of the conformal bootstrap program with some insights into the construction and properties of the bootstrap equation which involve fundamental aspects of crossing symmetry (OPE associativity) and unitarity.
Consider a metric \( g_{\mu\nu} \) in a \( d \) dimensional space time. The conformal transformations are defined as a set of diffeomorphism transformations that leave the metric unchanged up to an overall scale factor that may depend on the coordinates. Symbolically,

\[
    g'_{\mu\nu}(x') = \partial x^\rho \partial x'^\mu g_{\rho\sigma}(x) = \Lambda(x) g_{\mu\nu}(x) .
\]

Writing the infinitesimal transformations as \( x'^\mu = x^\mu + \epsilon^\mu \) and \( \Lambda = 1 - O(\epsilon) \), we obtain a set of constraints on the conformal killing vectors \( \epsilon_\mu \) as,

\[
    \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} (\partial^\rho \epsilon_\rho) ,
\]

where for simplicity, we have assumed that the conformal transformations are infinitesimal deformations of the standard Cartesian metric \( g_{\mu\nu} = \eta_{\mu\nu} \) with \( \eta_{\mu\nu} = \text{diag}(1,1,\cdots,1) \). Acting on (2.7) by \( \partial_\rho \) and permuting indices, we obtain,

\[
    \partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\mu \epsilon_\rho = \frac{2}{d} g_{\nu\rho} \partial_\mu (\partial^\sigma \epsilon_\sigma) ,
\]

\[
    \partial_\rho \partial_\mu \epsilon_\nu + \partial_\nu \partial_\rho \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial_\rho (\partial^\sigma \epsilon_\sigma) ,
\]

\[
    \partial_\nu \partial_\rho \epsilon_\mu + \partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d} g_{\mu\rho} \partial_\nu (\partial^\sigma \epsilon_\sigma) .
\]

Adding the first and third equation and subtracting off the second one gives,

\[
    \partial_\rho \partial_\nu \epsilon_\mu = \frac{1}{d} \left( g_{\mu\rho} \partial_\nu - g_{\nu\rho} \partial_\mu + g_{\mu\nu} \partial_\rho \right) (\partial^\sigma \epsilon_\sigma) .
\]

Finally contracting the indices with \( g^{\rho\nu} \) one obtains,

\[
    \Box \epsilon_\mu = \frac{2 - d}{d} \partial_\mu (\partial^\rho \epsilon_\rho) .
\]

Acting on the above equation by \( \partial_\nu \) gives,

\[
    \partial_\nu \Box \epsilon_\mu = \frac{2 - d}{d} \partial_\nu \partial_\mu (\partial^\rho \epsilon_\rho) ,
\]

while acting \( \Box \) on (2.7) gives,

\[
    \partial_\mu \Box \epsilon_\nu + \partial_\nu \Box \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \Box (\partial^\rho \epsilon_\rho) .
\]

Symmetrizing (2.11) and comparing with (2.12) gives,

\[
    (2 - d) \partial_\mu \partial_\nu f(x) = g_{\mu\nu} \Box f(x) ,
\]
where \( f(x) = \partial^\sigma \epsilon_\sigma \). Contracting with \( g^{\mu\nu} \) yields,

\[
(d - 1) \Box f(x) = 0.
\]  

(2.14)

For general \( d > 1 \), the above equation implies that the conformal killing vectors \( \epsilon_\mu \) must take the general form,

\[
\epsilon_\mu = c_\mu + a_{\mu\nu} x^\nu + b_{\mu\rho\sigma} x^\nu x^\rho.
\]  

(2.15)

From (2.9), we get,

\[
b_{\mu\rho\sigma} = \frac{1}{d} (g_{\mu\rho} b_\sigma + g_{\mu\sigma} b_\rho - g_{\rho\sigma} b_\mu),
\]  

(2.16)

where \( b_\mu = b^\sigma_\mu \). Again from (2.7), one obtains a constraint on the symmetric part of \( a_{\mu\nu} \) while the antisymmetric part remains unconstrained. The symmetric part is just proportional to the metric \( g_{\mu\nu} \) given by,

\[
a_{\{\mu\nu\}} = \frac{2}{d} g_{\mu\nu} \left( a_\sigma + \frac{1}{d} b_\rho x^\rho \right).
\]  

(2.17)

Counting the number of parameters appearing in the definition of \( \epsilon_\mu \), we find that,

\[
c : d, \ a_{\mu\nu} : \frac{d(d - 1)}{2} + 1, \ b_{\mu\nu\rho} : d.
\]  

(2.18)

The factor of 1 in \( a_{\mu\nu} \) comes from the symmetric part in \( a_{\mu\nu} \) proportional to the metric. Explicitly, the transformations are,

\[
\begin{align*}
 x'^\mu &= x^\mu + c^\mu, \quad \text{Translations} \\
 x'^\mu &= x^\mu + \lambda x^\mu, \quad \text{Dilatations} \\
 x'^\mu &= x^\mu + w_\mu^\nu x^\nu, \quad \text{Lorentz} \\
 x'^\mu &= x^\mu + 2(b_\sigma x^\sigma) x^\mu - x^2 b^\mu, \quad \text{SCT}
\end{align*}
\]  

(2.19)

where we have absorbed a factor of \( 1/d \) in the definition of \( b_\mu \). The finite form of the conformal transformations are obtained by exponentiation of the generators of the transformation. We will show one bit of the finite conformal transformation through the SCT (Special Conformal Transformation). This is generated by the following sequence of actions on the coordinates \( I p_\mu I \) where \( I \) stands for inversion and the operator reads (from right to left), inversion followed by translation and again followed by inversion. The inversion operator acting on the space time coordinate is given by,

\[
I : x_\mu = \frac{x_\mu}{x^2}.
\]  

(2.20)

Thus,

\[
x'^\mu = (I p_\mu I) x_\mu = I \left( \frac{x_\mu}{x^2} + b_\mu \right) = \frac{x_\mu}{x^2} + b_\mu + 2b \cdot x + b^2 x^2.
\]  

(2.21)
Following this, the conformal transformations on an arbitrary function $f(x)$ of the coordinates is,

$$
\begin{align*}
  f(x') &= f(x + \epsilon) = f(x) - e^\mu \partial_\mu f, \Rightarrow P_\mu = -i \partial_\mu. \\
  f(x') &= f(x + \epsilon) = f(x) - w^{\mu\nu} x_\mu \partial_\nu f, \Rightarrow M_{\mu\nu} = -i (x_\mu \partial_\nu - x_\nu \partial_\mu). \\
  f(x') &= f(x + \epsilon) = f(x) - \lambda^{\mu} \partial_\mu f, \Rightarrow D = -ix^{\mu} \partial_\mu. \\
  f(x') &= f(x + \epsilon) = f(x) + b^{\mu}(2x_\mu (x^{\rho} \partial_\rho) - x^2 \partial_\mu) f, \Rightarrow K_\mu = 2x_\mu (x^{\rho} \partial_\rho) - x^2 \partial_\mu,
\end{align*}
$$

which generates the generators for the conformal transformations. In place of $\epsilon_\mu$ we put the explicit form of the transformations acting on the function $f(x)$.

### 2.2.1 Conformal Algebra

The generators of the infinitesimal conformal transformations given in (2.22) satisfy certain commutation relations that generate the algebra of the conformal group. These commutations which we list below can be easily verified by their action on an arbitrary scalar function $f(x)$ of the coordinates.

\[
\begin{align*}
  [M_{\mu\nu}, P_\rho] &= i(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu), \\
  [M_{\mu\nu}, K_\rho] &= i(\delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu), \\
  [M_{\mu\nu}, M_{\rho\sigma}] &= i(\delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\rho} - \delta_{\mu\sigma} M_{\nu\rho}), \\
  [D, P_\mu] &= iP_\mu, \\
  [D, K_\mu] &= -iK_\mu, \\
  [K_\mu, P_\nu] &= -2i(\delta_{\mu\nu} D - M_{\mu\nu}).
\end{align*}
\]

The underlying group structure for the above algebra is $SO(d, 2)$ with $(d+1)(d+2)/2$ generators for a $d$ dimensional CFT.

### 2.3 States in CFT

We will now identify the states needed to define the Hilbert space of the conformal field theory. Notice that in a CFT it is possible to construct the Hilbert space ($\mathcal{H}$) that do not rely on the specifics of the theory but just on the element of conformal invariance. In a CFT, we foliate space time as spheres around the origin such that the states evolve from the smaller spheres to the larger ones due to the action of the dilatation operator. This is called Radial Quantization. Thus the field configurations on a certain sphere $S_{d-1}$ constitute the Hilbert space of those states on the boundary of $S_{d-1}$. For example, with no operator insertions, the vacuum $|0\rangle \in \mathcal{H}$ while with an operator insertion inside $S_{d-1}$, one can get,

\[
O(x)|0\rangle = |O(x)\rangle \in \mathcal{H}.
\]
Thus,

\[ \text{Operator}, \ \mathcal{O}(x) \Rightarrow \text{State}, \ |\mathcal{O}(x)\rangle. \quad (2.25) \]

To see the logic backwards, a given state \(|\psi\rangle \in \mathcal{H}\) can be decomposed into the basis of local eigenstates of the dilatation operator, on the boundary of \(S_{d-1}\) as follows.

\[ |\psi\rangle = |O_1\rangle + |O_2\rangle + \cdots, \ D|O_i\rangle = i\Delta_i|O_i\rangle, \quad (2.26) \]

which can themselves be thought of as the set of local operators \(O_i \in \mathcal{H}\) insertions in \(S_{d-1}\). Thus here,

\[ \text{State}, \ |\psi\rangle \Rightarrow \text{Operator}, \ O_i. \quad (2.27) \]

Thus the state-operator correspondence is stated as,

\[ \text{State} \ |\mathcal{O}(x)\rangle \leftrightarrow \text{Operator} \ \mathcal{O}(x) \quad (2.28) \]

\[ \text{local space of operators} \equiv \text{local space of states in the Radial Quantization} \quad (2.29) \]

### 2.3.1 Conformal transformations on \(\mathcal{O}(x)\)

Given this correspondence, it is now clear that the action of the conformal transformations on the space of local states is equivalent to that on the space of local operators acting on the Hilbert space. Henceforth we will consider scalar operators for convenience, though this discussion is equally applicable for the spin cases as well. Under the conformal map,

\[ g : x' = gx, \quad (2.30) \]

the operators \(\mathcal{O}(x)\) transform under the finite representations \(\mathcal{U}_g\), of the conformal transformation as,

\[ g : \mathcal{O}(x) \rightarrow \mathcal{O}'(x) = \mathcal{U}_g \mathcal{O}(x) \mathcal{U}^{-1}_g, \quad (2.31) \]

where \(\mathcal{U}_g\) forms the representation of the conformal group \(g\). The constraints due to the conformal symmetry are easily formulated in terms of the Euclidean correlation functions of the operators, which are also called the Wightmann Functions given by,

\[ W(x_1, \cdots, x_n) = \langle 0|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|0\rangle. \quad (2.32) \]

Since vacuum is invariant under the conformal transformations \(\mathcal{U}_g|0\rangle = |0\rangle\), we have, as the consequence of the conformal transformation,

\[ W'(x_1 \cdots, x_n) = \langle 0|\mathcal{O}'(x_1)\cdots\mathcal{O}'(x_n)|0\rangle = \langle 0|\mathcal{U}_g \mathcal{O}(x_1) \mathcal{U}_g^{-1} \cdots \mathcal{U}_g \mathcal{O}(x_n) \mathcal{U}_g^{-1}|0\rangle \]

\[ = \langle 0|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|0\rangle, \quad (2.33) \]

\[ = W(x_1, \cdots x_n). \]
2.3.1.1 Infinitesimal transformation: Primary+Descendants

Since the finite conformal transformations are exponentiation of the generators of the conformal group, viz.

\[ U_g = \exp(-i\epsilon_k L_k) \]  

The action of the infinitesimal transformation on the operator \( O(x) \) is given by,

\[ O'(x) = U_g O(x) U_g^{-1} = O(x) - i\epsilon_k [L_k, O(x)] \]  

where the Einstein summation convention is implied. Thus,

\[ \delta_k O(x) = -i [L_k, O(x)] \].  

For scale invariant theories, we diagonalize the dilatation operator, which acting on the operators at the origin give,

\[ [D, O(0)] = i\Delta O(0) \Rightarrow D|O(0)\rangle = i\Delta|O(0)\rangle \]  

Since the generators for the conformal transformations act as lowering operators, we further have,

\[ DK_\mu O(0)|0\rangle = ([D, K_\mu] + K_\mu D)O(0)|0\rangle = i(\Delta - 1)O(0)|0\rangle \].  

If the operators are annihilated by the generators of the special conformal transformation as given below,

\[ [K_\mu, O(0)] = 0 \Rightarrow K_\mu|O(0)\rangle = 0 \],

then we call them Primary Operators. In the above, we have consider the simplest case of Primary Scalar Operators. In addition if the operator carries spin, then the commutation relation with \( M_{\mu\nu} \) at \( x = 0 \) yields,

\[ [M_{\mu\nu}, O(0)] = S_{\mu\nu} O(0) \Rightarrow M_{\mu\nu}|O(0)\rangle = S_{\mu\nu}|O(0)\rangle \],

where \( S_{\mu\nu} \) is the intrinsic spin of the operator. Thus generic primary operators need dimension and spin to define the operator. All the operators which are annihilated by the generator of the special conformal transformation are called Primary Operators.

The descendants of these primary operators are generated by acting with the momentum generator on the primary operators. From the conformal algebra in (2.23), it appears that \( P_\mu \) acts as the raising operator while \( K_\mu \) acts as the lowering operator. Hence for any primary operator of dimension \( \Delta \), the descendants are given by,

\[ |O\rangle, \ P_\mu |O\rangle, \ P_\mu P_{\nu} |O\rangle, \cdots, P_{\mu_1} \cdots P_{\mu_n} |O\rangle \],

with dimensions \( \Delta, \ \Delta + 1, \ \Delta + 2, \cdots, \ \Delta + n \).
To see this, we act the dilatation operator on the first descendant,

\[ DP_\mu O(0)|0\rangle = ([D, P_\mu] + P_\mu D)O(0)|0\rangle = i(\Delta + 1)O(0)|0\rangle. \tag{2.42} \]

Similarly, by translating the dilatation operator through \( P_\mu \)s and using the commutation relation at each stage, one can arrive at the eigenvalue equation,

\[ DP_\mu_1 P_\mu_2 \cdots P_\mu_n O(0)|0\rangle = i(\Delta + n)P_\mu_1 P_\mu_2 \cdots P_\mu_n O(0)|0\rangle. \tag{2.43} \]

Any operator at a finite distance \( x \) from the origin, can be thought of as a linear combination of primary and descendant states at the origin. For example,

\[ O(x) = e^{-iP \cdot x} O(0) \rightarrow |O(x)\rangle = e^{-iP \cdot x} |O(0)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (-iP \cdot x)^n |O(0)\rangle, \tag{2.44} \]

since \( e^{iP \cdot x}|0\rangle = |0\rangle \) as the vacuum in invariant.

Using the transformations in (2.36) and for,

\[ \epsilon_k = \{ d^\mu, w^{\mu\nu}, \lambda, b^\mu \}, \quad L_k = \{ P_\mu, M_{\mu\nu}, D, K_{\mu} \}, \tag{2.45} \]

we get, for a generic operator \( O(x) \)

\[ [L_k, O(x)]|0\rangle = L_k e^{-iP \cdot x} O(0)|0\rangle = e^{-iP \cdot x}(e^{iP \cdot x} L_k e^{-iP \cdot x})O(0)|0\rangle, \tag{2.46} \]

and further using the identification of the generators from (2.22), we can write,

\[ [P_\mu, O(x)] = i\partial_\mu O(x), \quad [D, O(x)] = i(\Delta + x^\mu \partial_\mu)O(x), \]
\[ [K_{\mu}, O(x)] = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - x_\mu \Delta)O(x), \tag{2.47} \]
\[ [M_{\mu\nu}, O(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu + S_{\mu\nu})O(x). \]

### 2.3.1.2 Infinitesimal variation of Correlation functions

Furthermore the same symmetries will also show up in the constraints for the correlation functions. We will deal with the general \( n \)-point Wightmann functions whose variation under the infinitesimal transformations due to (2.36) are given by,

\[ \sum_{r=1}^{m} \delta_k W(r_1, \cdots, r_m) = 0, \tag{2.48} \]
\[ \Rightarrow \sum_{r=1}^{m} \langle 0| O_1(x_1) \cdots [L_k, O_r(x_r)] \cdots O_m(x_m)|0\rangle = 0. \]
We will assume that the correlation functions have the Poincare symmetry and will only consider the dilatation and the special conformal transformation. The Poincare symmetry already guarantees that the correlation function will have arguments of the form $|x_i - x_j|$ since only these quantities are Poincare invariant. Furthermore the dilatation and the special conformal generators impose additional restrictions on the correlation functions as,

$$m \sum_{r=1}^m (x^\mu_r \partial_{\mu r} + \Delta_r) \langle 0 | O_1 \cdots O_m | 0 \rangle = 0,$$

$$m \sum_{r=1}^m (x^2_r \partial^2_{\mu r} - 2(x_r)_\mu x^\nu_r \partial_{\nu r} - 2(x_r)_\mu \Delta_r) \langle 0 | O_1 \cdots O_m | 0 \rangle = 0.$$  \hspace{1cm} (2.49)

One can see that the general conformally invariant $n-$point functions will have to satisfy these constraint equations by considering just the case of the two point functions.

### 2.3.2 Conformal symmetry conditions: Wightmann functions

The $n-$point conformally invariant functions or the Wightmann functions can also be used to extract certain general features about the correlation functions if not the exact form. For example we can ask about the conformally invariant quantities defining a higher point function (say the four point function). Again we will assume that the correlation functions are Poincare invariant and consider the dilatation and the conformal inversions $R$, for the scalar operators of dimension $\Delta$, given by,

- **Dilatation**: $\mathcal{O}(x) \rightarrow \mathcal{O}'(x) = \lambda^\Delta \mathcal{O}(\lambda x)$,
- **Inversion**: $\mathcal{O}(x) \rightarrow \mathcal{O}'(x) = \frac{1}{(x^2)^\Delta} \mathcal{O}(Rx)$,  \hspace{1cm} (2.50)

where $R$ is the inversion. The Wightmann functions are defined as,

$$W(x_1, \cdots x_n) = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle.$$  \hspace{1cm} (2.51)

The conformal symmetry condition on the above correlation function implies that for $\mathcal{O}(x) \rightarrow \mathcal{O}'(x)$,

$$W(x_1, \cdots x_n) = W'(x_1, \cdots x_n).$$  \hspace{1cm} (2.52)

Due to the symmetry conditions in (2.50), the above equality translates into the following conditions for the correlation functions,

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \lambda^{\Delta_1+\Delta_2+\cdots} \langle \mathcal{O}_1(\lambda x_1) \cdots \mathcal{O}_n(\lambda x_n) \rangle,$$

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{1}{(x_1^2)_{\Delta_1}(x_2^2)_{\Delta_2} \cdots} \langle \mathcal{O}_1(Rx_1) \cdots \mathcal{O}_n(Rx_n) \rangle.$$  \hspace{1cm} (2.53)

We will consider each case separately when $n$ is either even or odd.
2.3.2.1 \( n \) is even

For even number of fields and coincident consecutive fields \( \mathcal{O}_i = \mathcal{O}_{i+1} \) for \( i = 1, \ldots, 2n-1 \), we have \( n \) independent fields \( \mathcal{O}_1, \ldots, \mathcal{O}_n \) with dimensions \( \Delta_1, \ldots, \Delta_n \) so that,

\[
\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2) \cdots \mathcal{O}_n(x_{2n-1})\mathcal{O}_n(x_{2n}) \rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_2} \cdots (x_{2n-1,2n}^2)^{\Delta_n}} F_n(x_1, \ldots, x_{2n}),
\]

where \( F_n(x_1, \ldots, x_n) \) is a function of dimensionless ratios of the square of coordinate differences \( x_{ij} = |x_i - x_j| \). The fact that the correlator is a function of \( x_{ij} \) is guaranteed from general Poincare invariance. Since the entire coordinate dependence is pulled out in the overall factor, the remaining function \( F_n \) under the transformations in (2.50) implies,

\[
F_n(x_1, \ldots, x_n) = F_n(Rx_1, \ldots, Rx_n).
\]

Since \( F_n \) is a function of \( x_{ij} \),

\[
Rx_{ij}^2 = (Rx_i - Rx_j)^2 = \left( \frac{1}{x_i} - \frac{1}{x_j} \right)^2 = \frac{x_{ij}^2}{x_i^2x_j^2}.
\]

Thus for \( F_n \) to be invariant under the conformal transformations in (2.50), the arguments of \( F_n \) should be of the form,

\[
u_n = \frac{x_{ij}^2x_{kl}^2}{x_{ik}^2x_{jl}^2}, 1 \geq i, j, k, l \geq 2n,
\]

so that,

\[
\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2) \cdots \mathcal{O}_n(x_{2n-1})\mathcal{O}_n(x_{2n}) \rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_2} \cdots (x_{2n-1,2n}^2)^{\Delta_n}} F_n(u_1, u_2 \cdots u_n),
\]

where \( u_n \)'s are called the \textit{conformal cross ratios}. For concreteness we will consider the examples of the two point functions and the four point functions below using the above arguments.

\[\textbullet\ 2\text{-pt function}\]

Consider the case when we have only two fields \( \mathcal{O}_1(x_1) \) and \( \mathcal{O}_1(x_2) \) with conformal dimension \( \Delta_1 \). The correlator (Wightmann function) is given by,

\[
W(x_1, x_2) = \langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2) \rangle = \frac{1}{(x_{12}^2)^{\Delta_1}} F_1(x_1, x_2).
\]

Invariance of the Wightmann function under the conformal transformations in (2.50) implies,

\[
F_1(x_1, x_2) = F_1(Rx_1, Rx_2).
\]
But we know that the only invariant function $F_1$ for this correlator satisfying (2.50) is

$$F_1(x_1, x_2) = \text{constant}.$$  \hspace{1cm} (2.61)

Thus,

$$W(x_1, x_2) = \langle O_1(x_1)O_1(x_2) \rangle = \frac{\text{constant}}{(x_{12}^2)^{\Delta_1}},$$  \hspace{1cm} (2.62)

which fixes the form of the two point function up to an overall constant. As one can check, this form satisfies the symmetry conditions of the Wightmann functions in (2.49).

- **4-pt function**

This case is a little bit non trivial but the same arguments can be applied here as well. To start with, consider the Wightmann function for four fields $O_1(x_1), O_1(x_2), O_2(x_3)$ and $O_2(x_4)$ with conformal dimensions $\Delta_1, \Delta_1, \Delta_2$ and $\Delta_2$. A detailed discussion on the explicit form of the four point function will be given later but even without that, it is easy to see the symmetry structure of the above four point function by considering,

$$W(x_1, x_2, x_3, x_4) = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_2}} F_2(x_1, x_2, x_3, x_4).$$  \hspace{1cm} (2.63)

Again using the same arguments,

$$F_2(x_1, x_2, x_3, x_4) = F_2(Rx_1, Rx_2, Rx_3, Rx_4),$$  \hspace{1cm} (2.64)

it is clear that $F_2$ should be a function of two independent variables,

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u, \quad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v,$$  \hspace{1cm} (2.65)

so that,

$$W(x_1, x_2, x_3, x_4) = \langle O_1(x_1)O_1(x_2)O_2(x_3)O_2(x_4) \rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_2}} F_2(u, v).$$  \hspace{1cm} (2.66)

As a special case, we would like to comment on the case when all the external scalars are identical. The interchange of the coordinates $x_i$ do not produce different results imposing further restrictions on the form of the functions $F_2(u, v)$, since,

$$W(x_1, x_2, x_3, x_4) = W(x_1, x_4, x_3, x_2) = W(x_1, x_3, x_2, x_4) = \cdots.$$  \hspace{1cm} (2.67)

A more general case when all the external scalars are different or at least some are different and a more detailed discussion on the form of $F_2$ will not be presented in this thesis.
2.3.2.2 \( n \) is odd: 3-point function

Having discussed the form of the correlators in some details for the case of even number of fields, we will now consider a preliminary example of the case when \( n \) (the number of fields) is odd. For this case, the correlator (Wightmann function) cannot be written in the form of overall scale factors times and conformally invariant function of the cross ratios. A simple example of the three point function will clarify this issue. Consider the correlator of three fields \( O_1(x_1), O_2(x_2) \) and \( O_3(x_3) \) with conformal dimensions \( \Delta_1, \Delta_2 \) and \( \Delta_3 \). This is given by,

\[
W(x_1, x_2, x_3) = \langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle \sim \frac{1}{(x_{12}^2)^{\delta_1}(x_{23}^2)^{\delta_2}(x_{13}^2)^{\delta_3}}
\]  

(2.68)

The transformations in (2.50) imply,

\[
\lambda^{\Delta_1+\Delta_2+\Delta_3}\langle O_1(\lambda x_1)O_2(\lambda x_2)O_3(\lambda x_3) \rangle = \langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle, \\
\frac{1}{(x_1^2)^{\Delta_1}(x_2^2)^{\Delta_2}(x_3^2)^{\Delta_3}}\langle O_1(Rx_1)O_2(Rx_2)O_3(Rx_3) \rangle = \langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle.
\]  

(2.69)

The first condition gives \( \delta_1 + \delta_2 + \delta_3 = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3) \) while the second condition gives,

\[
\delta_1 + \delta_2 = \Delta_3, \quad \delta_2 + \delta_3 = \Delta_1, \quad \delta_1 + \delta_3 = \Delta_2,
\]  

(2.70)

whereby,

\[
\delta_1 = \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1), \quad \delta_2 = \frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2), \quad \delta_3 = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3).
\]  

(2.71)

Thus finally,

\[
W(x_1, x_2, x_3) \sim \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}(x_{23}^2)^{\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)}(x_{13}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}},
\]  

(2.72)

which fixes the three point function up to an overall constant.

2.4 Tensor fields

We will give a brief description of the tensor operators in CFT and conformal transformations acting on them, in this section. We will mainly follow [1] in this section. A generic tensor field \( \Phi_{\Delta,j} \) is labeled by three quantum numbers \( \Delta, j_1, j_2 \) of which the last two are the Lorentz indices. By construction we have chosen Poincare symmetry to be inbuilt in the operators. Henceforth, we will consider the special case of the tensor operators for symmetric-traceless operators where \( j_1 = j_2 = \ell/2 \). To find the invariant functions in the Euclidean signature, we will need the action of the dilatation and the conformal inversion \( R \), on the tensor operators, which take the
The same argument carries forward for the higher even point functions as well. Similarly we also, we can extend the analysis for higher odd point correlators.

Chapter 2. Conformal Field Theory

...operators are given by,

\[ W \]

is the factor accompanying the inversion operation. The two point functions of the tensor operators are given by,

\[ W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell}(x_{12}) = \langle \Phi_{\mu_1, \ldots, \mu_k}^{\Delta, \ell}(x_1) \Phi_{\nu_1, \ldots, \nu_k}^{\Delta, \ell}(x_2) \rangle. \] (2.75)

Similar to the case of scalar operators in (2.53), we can also formulate the conformal invariance on the correlators of the tensor operators with slight deformations as,

\[ W_{\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k}^{\Delta, \ell}(x_{12}) = \lambda^{2\Delta} W_{\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k}^{\Delta, \ell}(\lambda x_{12}), \]

\[ W_{\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k}^{\Delta, \ell}(x_{12}) = (\frac{x^{12}}{x^{12}})^{-\Delta} g_{\mu_1 \nu_1}(x_1) \cdots g_{\mu_k \nu_k}(x_1) g_{\nu_1 \tau_1}(x_2) \cdots g_{\nu_k \tau_k}(x_2) W_{\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k}^{\Delta, \ell}(Rx_{12}), \] (2.76)

where we will use the following properties of \( g_{\mu \nu}(x) \),

\[ g_{\mu}^{\rho}(x)g_{\mu \nu}(x) = \delta_{\mu \nu} + \frac{x_{\mu} x_{\nu}}{x^2}, \] (2.77)

For the case of symmetric-traceless tensors,

\[ W_{\mu_1, \ldots, \nu_1}^{\Delta, \ell}(x_{12}) \sim (g_{\mu_1 \nu_1}(x_{12}) \cdots g_{\mu_k \nu_k}(x_{12}) - \text{traces}(x_{12}^{\Delta})). \] (2.78)

The same argument carries forward for the higher even point functions as well. Similarly we can consider the case of the three point functions given by,

\[ W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell, \Delta_2, \Delta_3}(x_1, x_2, x_3) = \langle \Phi_{\mu_1, \ldots, \mu_k}^{\Delta, \ell}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta_3}(x_3) \rangle, \] (2.79)

for which the constraints from the dilatation and the inversion take the form,

\[ W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell, \Delta_2, \Delta_3}(x_1, x_2, x_3) = \lambda^{\Delta + \Delta_2 + \Delta_3} W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell, \Delta_2, \Delta_3}(\lambda x_1, \lambda x_2, \lambda x_3), \]

\[ W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell, \Delta_2, \Delta_3}(x_1, x_2, x_3) = (\frac{x_1}{x_3})^{-\Delta} (\frac{x_2}{x_3})^{-\Delta_2} (\frac{x_3}{x_1})^{-\Delta_3} g_{\mu_1 \nu_1}(x_1) \cdots g_{\mu_k \nu_k}(x_1) W_{\mu_1, \ldots, \mu_k}^{\Delta, \ell, \Delta_2, \Delta_3}(Rx_1, Rx_2, Rx_3), \] (2.80)

from which one can derive the form of the three point functions up to overall constants. Here also, we can extend the analysis for higher odd point correlators.
2.5 Operator Product Expansion

We will focus on the generic properties of the Hilbert space for any conformal field theory that do not rely on the specifics of the theory. Needless to say, the generic features will be constrained by conformal symmetry. In general the Hilbert space contains multiparticle states of the form,

\[ \mathcal{O}(x)|0\rangle, \mathcal{O}(x)\mathcal{O}(y)|0\rangle, \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z)|0\rangle, \cdots, \] (2.81)

where \(|0\rangle\) is the conformally invariant vacuum. Consider a set of basis states,

\[ |\mathcal{O}_\Delta(x)\rangle = \mathcal{O}_\Delta(x)|0\rangle. \] (2.82)

We can define the Hilbert space in terms of the Wightmann functions which acts as the Kernel of invariant scalar product. Specifically,

\[ W(x_1, x_2) = \langle 0|\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)|0\rangle, \] (2.83)

so that any state \(|f\rangle\) can be expanded in terms of the basis states \(|\mathcal{O}_\Delta(x)\rangle\) of the Hilbert space. We write the functions following [1], as,

\[ f_\Delta(x) = \langle f|\mathcal{O}_\Delta(x)\rangle, \] (2.84)

so that, the Kernel of the invariant scalar function can be represented as,

\[
\langle f|f\rangle = \int d^d x d^d y \langle f|\mathcal{O}_\Delta(x)\rangle W^{-1}_\Delta(x, y)\langle\mathcal{O}_\Delta(y)|f\rangle, \\
= \int d^d x d^d y f_\Delta(x)W^{-1}_\Delta(x, y)f_\Delta^\ast(y). \] (2.85)

We also note that,

\[
\int d^d x W_\sigma(x_1 - x)W_\sigma^{-1}(x - x_2) = \delta(x_1 - x_2). \] (2.86)

In contrary, we can define another set of non-local operators, given by,

\[ |\tilde{\mathcal{O}}(x)\rangle = \int d^d y W_{\mathcal{O}}^{-1}(y - x)|\mathcal{O}(y)\rangle, \] (2.87)

for which we can define the corresponding correlator as,

\[ \langle \tilde{\mathcal{O}}(x)\tilde{\mathcal{O}}(y)\rangle = W_{\tilde{\mathcal{O}}}(x, y), \] (2.88)
Using this definition it is easy to see that,

\[
W_{\mathcal{O}}(x_1, x_2) = \langle 0 | \hat{\mathcal{O}}(x_1) \hat{\mathcal{O}}(x_2) | 0 \rangle = \int dy_1 \, dy_2 \, W_{\mathcal{O}}^{-1}(x_1, y_1) W_{\mathcal{O}}^{-1}(y_2, x_2) \langle 0 | \mathcal{O}(y_1) \mathcal{O}(y_2) | 0 \rangle \\
= \int dy_1 \, dy_2 \, W_{\mathcal{O}}^{-1}(x_1, y_1) W_{\mathcal{O}}^{-1}(y_2, x_2) W_{\mathcal{O}}(y_1, y_2) ,
\]

(2.89)

More generally, for any two operators, we can write in terms of the basis elements \(|\mathcal{O}(x)\rangle\),

\[
W(x_1, x_2) = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \int dx \, dy \, A_{\mathcal{O}}^2 \langle \mathcal{O}_1(x_1) \mathcal{O}(x) \rangle W_{\mathcal{O}}^{-1}(x, y) \langle \mathcal{O}(y) \mathcal{O}_2(x_2) \rangle ,
\]

(2.90)

Thus we can write,

\[
|\mathcal{O}\rangle = \int dx \, dy \, A_{\mathcal{O}}^2 |\mathcal{O}(x)\rangle W_{\mathcal{O}}^{-1}(x, y) \langle \mathcal{O}(y) | = \int d^d x \, A_{\mathcal{O}}^2 |\hat{\mathcal{O}}(x)\rangle \langle \mathcal{O}(x) | ,
\]

(2.91)

as the analog of the Completeness condition (more compact in terms of the \(|\hat{\mathcal{O}}(x)\rangle\) basis) for the theory and \(A_{\mathcal{O}}^2\) is the (square of the OPE) coefficient associated with the exchange \(\mathcal{O}\). More generally over a set of all basis states spanning the Hilbert space \((\mathcal{H})\) of the theory,

\[
|\mathcal{O}\rangle = \sum_{\mathcal{O}_i} \int dx \, dy \, A_{\mathcal{O}_i}^2 |\mathcal{O}_i(x)\rangle W_{\mathcal{O}_i}^{-1}(x, y) \langle \mathcal{O}_i(y) | ,
\]

(2.92)

where \(A_{\mathcal{O}_i}^2\) are the (squares of the OPE) coefficients attached with a particular operator exchange \(\mathcal{O}_i\), is the full Completeness condition for \(\mathcal{H}\). Note that in the above definition, there is no restriction over \(i\) in the sense that it includes generic operators of the form \(\mathcal{O}_{\Delta, \ell}\) with conformal dimension \(\Delta\) and spin \(-\ell\) appearing in the spectrum.

Similarly one can consider the three point functions as well,

\[
\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \mathcal{O}_{\Delta}(x) \rangle = W_{\Delta, \Delta_1, \Delta_2}(x_1, x_2) .
\]

(2.93)

In terms of the basis elements \(|\hat{\mathcal{O}}_{\Delta}(x)\rangle\) (and using (2.91) modulo the overall factor of \(A_{\mathcal{O}}^2\)), we can write,

\[
Q_{\Delta, \Delta_1, \Delta_2}(x_1, x_2) = \langle \hat{\mathcal{O}}_{\Delta}(x) \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \int dy \, W_{\mathcal{O}}^{-1}(x, y) W_{\Delta, \Delta_1, \Delta_2}(y, x_1, x_2) .
\]

(2.94)

\(Q_{\Delta, \Delta_1, \Delta_2}(x_1, x_2)\) is the three point function in the \(\hat{\mathcal{O}}_{\Delta}(x)\) basis and we have used the completeness condition. Thus every two particle states given by \(\mathcal{O}(x)\mathcal{O}(y)|0\rangle\) can be written in terms of the basis elements \(|\mathcal{O}(x)\rangle\) as,

\[
\mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) |0\rangle = \int dy \, A_{\mathcal{O}} Q_{\Delta, \Delta_1, \Delta_2}(y, x_1, x_2) |\mathcal{O}(y)\rangle .
\]

(2.95)
Here we have included only one element of the $\mathcal{O}_\Delta(x)$ basis. In general,

$$
\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)|0\rangle = \sum_{\mathcal{O}_\ell} A_{\mathcal{O}_\ell} \int dy \, Q_{\Delta,\Delta_1,\Delta_2}(y, x_1, x_2)|\mathcal{O}_{\Delta,\ell}(y)\rangle ,
$$

(2.96)

where $A_{\mathcal{O}_\ell}$ is the corresponding OPE coefficient corresponding the exchange of the operator $\mathcal{O}_\ell$. In principle $\mathcal{O}_\ell$ can be operators with spin as well. For generic spin--$\ell$ operators we can similarly define the conformally invariant two point function as,

$$
W_{\Delta,\ell}(x_1, x_2) = \langle \mathcal{O}_{\Delta,\ell}(x_1)\mathcal{O}_{\Delta,\ell}(x_2)\rangle ,
$$

(2.97)

so that,

$$
A_{\mathcal{O}_\ell} Q_{\Delta,\Delta_1,\Delta_2}^{\ell}(x_1, x_2) = \int dy \, W_{\Delta,\ell}^{-1}(x, y)\langle \mathcal{O}_{\Delta,\ell}(y)\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(x)\rangle .
$$

(2.98)

Similarly for a generic $n$--point function ($n = m + 2$), we can define, after inserting the completeness condition in (2.92),

$$
\langle 0|\Phi_{i_1} \cdots \Phi_{i_m} \mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}|0\rangle = \sum_{\mathcal{O}_i} (A_{\mathcal{O}_i})^2 \int dx dy \, \langle 0|\Phi_{i_1} \cdots \Phi_{i_m}\mathcal{O}_i(x)|0\rangle
$$

$$
W_{\mathcal{O}_i}^{-1}(x, y)\langle 0|\mathcal{O}_i(y)\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(x)|0\rangle .
$$

(2.99)

This naturally leads to the operator version of the generic $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle$ which is given by,

$$
\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle = \sum_{\mathcal{O}_i} \int dx \, C_{12i}(x, x_1, x_2)\mathcal{O}_i(x)|0\rangle .
$$

(2.100)

or equivalently,

$$
\mathcal{O}_1(x_1) \times \mathcal{O}_2(x_2) = \sum_{\mathcal{O}_i} \int dx \, C_{12i}(x, x_1, x_2)\mathcal{O}_i(x) .
$$

(2.101)

The above operator relation is known as the Operator Product Expansion. For the simplest case when the external operators are scalars of dimensions $\Delta_1$ and $\Delta_2$ and the exchange is also a scalar of dimension $\Delta$, the function $C_{12i}(x, x_1, x_2)$ are given by [1],

$$
C_{12i}(x, x_1, x_2) = \frac{A_{12i}}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta)}} \left(1 + \frac{\Delta - \Delta_1 + \Delta_2}{2\Delta} x_{12}^\mu \partial_\mu
\right.
$$

$$
+ \frac{1}{8} \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta - \Delta_1 + \Delta_2 + 2)}{\Delta(\Delta + 1)} x_{12}^\mu x_{12}^\nu \partial_\mu \partial_\nu
$$

$$
- \frac{1}{16} \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta + \Delta_1 - \Delta_2)}{\Delta(\Delta + 1 - d/2)} x_{12}^\mu (x_{12}^2)\Box \partial_\mu + \cdots \right) \delta^4(x_{12}) ,
$$

(2.102)

where $x_{ij} = |x_i - x_j|$. This analysis can be extended for the case where we have general operators of conformal dimensions $\Delta$ and spin--$\ell$ appearing in the spectrum. Finally, consider the four point function of scalar operators,

$$
\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|0\rangle .
$$

(2.103)
inserting the generalized completeness condition in (2.92), in the above four point function between the operators $\mathcal{O}_2$ and $\mathcal{O}_3$, we can write the four point function as,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle = \sum_{\mathcal{O}_\ell} (A_{\mathcal{O}_\ell})^2 \int dx dy \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_\ell(x) \rangle \times W_{\ell}^{-1}(x,y)\langle \mathcal{O}_\ell(y)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle .$$

(2.104)

Note that each of the constituent of the four point function is a three point function connected by the propagator for the $\mathcal{O}_\ell$ operator between the points $x$ and $y$. The function in the integral has a familiar interpretation which is known as the Conformal Block $G_{\Delta,\ell}(u,v)$ where $u$ and $v$ are the conformal cross ratios given by,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

(2.105)

Writing this explicitly,

$$G_{\mathcal{O}_\ell}(u,v) = \int dx dy \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_\ell(x) \rangle W_{\ell}^{-1}(x,y)\langle \mathcal{O}_\ell(y)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle .$$

(2.106)

With the overall normalizations, we find that, the four point function for a particular exchange $\mathcal{O}_\ell$ as,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle_{\mathcal{O}_\ell} = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left( \frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_{12}}{2}} \left( \frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_{34}}{2}} P_{\mathcal{O}_\ell} G_{\mathcal{O}_\ell}(u,v).$$

(2.107)

so that, the total four point function becomes,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left( \frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_{12}}{2}} \left( \frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_{34}}{2}} \sum_{\mathcal{O}_\ell} P_{\mathcal{O}_\ell} G_{\mathcal{O}_\ell}(u,v),$$

(2.108)

where $\Delta_{ij} = \Delta_i - \Delta_j$, $P_{\mathcal{O}_\ell}$ is the OPE coefficient squared and $G_{\mathcal{O}_\ell}$ is the corresponding conformal block. The conformal block has the information about a particular primary operator and the set of infinite numbers of descendants from it therein.

### 2.5.1 Convergence of the OPE

Before moving on to the Conformal Bootstrap Program, we will pause here to comment on the convergence properties of the OPE in conformal field theories. For this section we will mainly follow [12]. An OPE of two scalars is given by,

$$\phi(x)\phi(0) \overset{x \to 0}{\approx} \sum_{\mathcal{O}} C(x^2)x^{\mu_1} \cdots x^{\mu_\ell}\mathcal{O}_{\mu_1 \cdots \mu_\ell}(0),$$

(2.109)
where $O_{\mu_1 \cdots \mu_\ell}$ in general contains scalars as well as symmetric-traceless operators of spin $\ell$. We have considered the simplest case of identical scalars, though the above arguments are equally valid for generic operators. The above expression has the following properties:

1. The function $C(x^2)$ is fixed by the dimensionality of the external scalars and the exchange operator. This is,
   \begin{equation}
   C(x^2) = \text{const.} (x^2)^{-\Delta_\phi + (\Delta_O - \ell)/2}.
   \end{equation}

2. The functions in the OPE are fixed by conformal invariance up to some overall constant factors.

   \begin{equation}
   \phi(x)\phi(y) \sim \sum_O f_{\phi\phi} O(x - y, \partial_y) O(y),
   \end{equation}

   where we have suppressed all the spin indices. The function $P(x - y, \partial_y)$ contains information about all the descendants and the coefficients appearing in the function are fixed by conformal invariance. Thus any $n$–pt function can be written as,

   \begin{equation}
   \langle \phi(x)\phi(y)\prod_i \psi(z_i) \rangle = \sum_O f_{\phi\phi} O(x - y, \partial_y)\langle O \prod_i \psi(z_i) \rangle.
   \end{equation}

3. Convergence: This means that any $n$–point function like (2.112) can be built by repeatedly using the OPE and is absolutely convergent for any finite separation $|x - y|$ rather than only the limiting case $x \to y$. This further implies that any higher point function can be calculated using the OPE provided, we know the operator content $O_i$ and the OPE coefficients $f_{ijk}$ of the spectrum. And one can use the OPE to calculate the four point functions in any channel (12)(34), (13)(24) and (14)(23).

As a simple proof of OPE convergence, following [13], we will give the arguments from radial quantization, which states that the OPE is absolutely convergent for any separation $|x - y|$ which is smaller than the least distance of any other operator insertion (say at $z_i$). Thus the OPE is convergent for,

\begin{equation}
|x - y| < \min_i |z_i - y|.
\end{equation}

To prove this, we can consider a sphere centered around $y$ as origin satisfying the above condition. We can then write (2.112) as a scalar product of two states $\langle \Psi|\Phi \rangle$ given by,

\begin{equation}
|\Phi\rangle = \phi(x)\phi(y)|0\rangle, \quad \langle \Psi| = \langle 0| \prod_i \psi(z_i).\end{equation}

Within the sphere and with no other operator insertion, we can decompose $|\Phi\rangle$ into an orthonormal basis of eigenstates of the radial Hamiltonian,

\begin{equation}
|\Phi\rangle = \sum_n C_n(x - y)|E_n\rangle.
\end{equation}
where $|E_n\rangle$ are the eigenstates of the Hamiltonian operator which is the dilatation operator in radial quantization. The OPE is in one-one correspondence with the above decomposition due to the state-operator correspondence in the radial quantization. The states $|E_n\rangle$ are made from the primaries of the Hamiltonian and the descendants therein,

$$|E_n\rangle = (\partial_y)^n \mathcal{O}(y)|0\rangle, \quad E_n = \Delta_{\mathcal{O}} + n,$$

and form an orthonormal basis of the Hilbert space. Now using a theorem about the Hilbert space: *Scalar product of two states converges when one of them can be decomposed in terms of an orthonormal basis* shows that the OPE is convergent for any finite $|x - y|$ satisfying (2.113).

### 2.6 Conformal Bootstrap

We can now turn to the following question:

"What are the necessary ingredients to determine the CFT completely?"

The answer to this question is the main content of this section. To have a complete knowledge of the CFT spectrum we need the following contents:

1. The knowledge about the operators entering the OPE of the CFT. This is precisely the information about various operators $\mathcal{O}_i$ labeled by the quantum numbers $\Delta$ and $\ell$ in the spectrum.

2. The couplings of these operators appearing in the spectrum. This information is encoded in the OPE coefficients corresponding to the specific operator exchange.

3. It should be possible to determine all the correlators in the CFT.

Thus to sum up, a complete knowledge of any conformal field theory includes the information about the full operator content of the theory, all the couplings appearing in the theory (reflected in the OPE coefficients) and determining all the correlators in the theory. Note that all this information is encoded in the four point function in (2.108). For simplicity, we will consider the four point function of identical external scalars with conformal dimensions $\Delta_{\phi}$, but the same treatment can also be extended to the case where the external operators are with spin. In case of identical scalars there is an additional simplification that the OPE consists of only symmetric-traceless objects of the generic form,

$$\mathcal{O}_{\Delta,\ell} \equiv (\phi \partial_{\mu_1} \cdots \partial_{\mu_4} (\partial^2)^n \phi)_{\text{sym}} - \text{traces},$$

thus implying that only even spin operators can be present in the spectrum. The subscript \text{sym} implies that the object is symmetrized with respect to the change of indices and we have subtracted off the trace parts to make this operator traceless.
The conformal block can be obtained by a brute force integration of (2.106) using the explicit form of the three point functions obtained from the OPE of identical scalars. Here we will focus on another way of deriving the conformal blocks from the Casimir differential equation satisfied by the blocks. This is best described in the embedding space where we embed the $d$ dimensional Minkowski space in the $d+2$ dimensional with the metric [1],

$$g_{AB} = \text{diag}\{+, -, \cdots, -, +\}, \quad (2.118)$$

and coordinates,

$$X^A, \ A = 0, 1, \cdots, d - 1, d + 1, d + 2. \quad (2.119)$$

The embedding space has the algebra,

$$SO(d, 2), \rightarrow X^A \rightarrow X'^A = \Lambda^A_B X^B, \quad \text{and} \quad \Lambda^C_A g_{CD} \Lambda^D_B = g_{AB}. \quad (2.120)$$

The $d$-dimensional Minkowski is the embedding in the directions of the isotropic light rays satisfying,

$$g_{AB} X^A X^B = 0, \Rightarrow X^2_0 - \sum_{i=1}^{d-1} X^2_i - X^2_{d+1} - X^2_{d+2} = 0. \quad (2.121)$$

One can thus define from this, the coordinates of the $d$ dimensional space as,

$$x_\mu = \frac{X_\mu}{\rho_+}, \quad x^2 = \frac{\rho_+ - \rho_-}{\rho_+}, \quad \rho_+ = X_{d+1} + X_{d+2}, \quad \rho_- = X_{d+1} - X_{d+2}, \quad \mu = 0, \cdots, d - 1. \quad (2.122)$$

Conformal transformations in the $d$ dimensional Minkowski space is given by pseudo-rotations in $d+2$ dimensional embedding space. For example,

1. **Lorentz transformations**

$$X_\mu \rightarrow X'_\mu = \lambda_\nu^\mu X^\nu, \quad \rho_\pm \rightarrow \rho'_\pm = \rho_\pm. \quad (2.123)$$

2. **Translations**

$$X_\mu \rightarrow X'_\mu = X_\mu + \rho_+ a_\mu, \quad \rho_+ \rightarrow \rho'_+ = \rho_+, \quad (2.124)$$

and for the remaining $\rho_-$ we have,

$$X_\mu X^\mu - \rho_- \rho_+ = 0 \Rightarrow X'_\mu X'^\mu - \rho'_- \rho'_+ = 0, \quad (2.125)$$

which gives,

$$\rho_- \rightarrow \rho'_- = \rho_- + 2 X_\mu a^\mu + \rho_+ a^2. \quad (2.126)$$

3. **Dilatations**

$$X_\mu \rightarrow X'_\mu = e^\lambda X_\mu, \quad \mu = 0, \cdots, d - 1. \quad (2.127)$$
This is generated by pseudo-rotations in the $d+1, d+2$ coordinates given by,

$$
X_{d+1} \rightarrow X'_{d+1} = X_{d+1} \sinh \lambda + X_{d+2} \cosh \lambda ,
X_{d+2} \rightarrow X'_{d+1} = X_{d+1} \cosh \lambda + X_{d+2} \sinh \lambda ,
$$

so that,

$$
\rho_\pm \rightarrow \rho'_\pm = e^{\pm \lambda} \rho_\pm .
$$

4. Special conformal transformations

$$
X_\mu \rightarrow X'_\mu = X_\mu + \rho_- a_\mu , \rho_- \rightarrow \rho'_- = \rho_- ,
$$

and finally,

$$
\rho_+ \rightarrow \rho'_+ = \rho_+ + 2X_\mu a^\mu + \rho_- a^2 .
$$

The $SO(d, 2)$ algebra for the embedding space is given by the commutation relations,

$$
[J_{AB}, J_{CD}] = -i(g_{AC}J_{BD} + g_{BD}J_{AC} - g_{AD}J_{BC} - g_{BC}J_{AD}) .
$$

where the differential representation of the generators $J_{AB}$ of the $SO(d, 2)$ group are given by,

$$
J_{AB} = i \left( X_A \frac{\partial}{\partial X_B} - X_B \frac{\partial}{\partial X_A} \right) , A, B = 0, \cdots , d-1, d+1, d+2 .
$$

More explicitly,

$$
J_{\mu \nu} = M_{\mu \nu} , P_\mu = J_{\mu, d+2} - J_{\mu, d+1} , K_\mu = J_{\mu, d+1} + J_{\mu, d+2} , D = J_{d+1, d+2} .
$$

The quadratic Casimir equation is given by,

$$
C_2 = \frac{1}{2} J_{AB} J_{AB} = \frac{1}{2} J^{\mu \nu} J_{\mu \nu} + J^{\mu, d+1} J_{\mu, d+1} - J^{\mu, d+2} J_{\mu, d+2} + J^{d+1, d+1} J_{d+1, d+2} ,
\equiv \frac{1}{2} M^{\mu \nu} M_{\mu \nu} - \frac{1}{2} (K^\mu P_\mu + P^\mu K_\mu) - D^2 .
$$

For a primary operator $O(x)$ transforming in the representation $(\Delta, j_1, j_2)$, the action of $C_2$ on the field yields,

$$
C_2 O(x) = [\Delta(\Delta - d) + 2(j_1(j_1 + 1) + j_2(j_2 + 1))] O(x) = E_{\Delta, j_1, j_2} O(x) ,
$$

which yields the eigenvalue for $O(x)$. For symmetric traceless operators $j_1 = j_2 = \ell/2$ and hence,

$$
E_{\Delta, \ell} = \Delta(\Delta - d) + \ell(\ell + 2) .
$$

We can also find the action of the Casimir operator $C_2$ on the OPE of two distinct operators. For simplicity, we will consider identical scalars for which only symmetric-traceless operators
appear in the OPE. Thus,
\[ C_2 \mathcal{O}(x) \mathcal{O}(y) |0\rangle = C_2 \sum_{\Delta, \ell} C_{\Delta, \ell} (\mathcal{O}_{\Delta, \ell} + \text{descendants}) |0\rangle, \]
\[ = \sum_{\Delta, \ell} C_{\Delta, \ell} C_2 (\mathcal{O}_{\Delta, \ell} + \text{descendants}) |0\rangle, \]
\[ = \sum_{\Delta, \ell} C_{\Delta, \ell} E_{\Delta, \ell} (\mathcal{O}_{\Delta, \ell} + \text{descendants}) |0\rangle. \]  

(2.138)

This can be used to determine the differential equation satisfied by the conformal blocks. For a particular exchange, the conformal block in the embedding space is given by [2],
\[ \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle_{\ell} = \frac{P_{\mathcal{O}_1 \mathcal{O}_2}(u, v)}{(X_1 \cdot X_2)^{\frac{1}{2} (\Delta_1 + \Delta_2)} (X_3 \cdot X_4)^{\frac{1}{2} (\Delta_3 + \Delta_4)}} \left( \frac{X_1 \cdot X_4}{X_1 \cdot X_3} \right)^{\Delta_1 \Delta_2} \left( \frac{X_2 \cdot X_4}{X_2 \cdot X_3} \right)^{\Delta_3 \Delta_4}, \]

where,
\[ x^2_{ij} = -X_i \cdot X_j, \text{ and } u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}, \quad v = \frac{(X_1 \cdot X_1)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}. \]  

(2.139)

in the embedding space coordinates $X^A$. From the OPE relation in (2.138), one can immediately see that the action of the Casimir operator on the four point function should also yield,
\[ C_2 \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle_{\ell} = E_{\Delta, \ell} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle_{\ell}. \]

(2.141)

Plugging in the form of the four point function in (2.107), and using the differential notation of $J_{AB}$ given in (2.133), we find that the conformal block for a particular exchange satisfies the differential equation,
\[ L^2 G_{\ell} - (\Delta_{12} - \Delta_{34}) \left( (1 + u - v) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) - (1 - u - v) \frac{\partial}{\partial v} \right) G_{\ell} \]
\[ - \frac{1}{2} \Delta_{12} \Delta_{34} G_{\ell} = -(\Delta (\Delta - d) + \ell (\ell + 2)) G_{\ell}, \]

where for any function $F(u, v)$ the differential operator acting on the function can be written as,
\[ \frac{1}{2} L^2 F = -((1 - v)^2 - u(1 + v)) \frac{\partial}{\partial v} v \frac{\partial}{\partial v} F - (1 - u + v) u \frac{\partial}{\partial u} u \frac{\partial}{\partial u} F \]
\[ + 2(1 + u - v) u v \frac{\partial^2}{\partial u \partial v} F + du \frac{\partial}{\partial u} F. \]  

(2.143)

Since the conformal block is labeled by two quantum numbers $\Delta$ and $\ell$, we will henceforth label the conformal block for $\mathcal{O}_\ell$ exchange as,
\[ G_{\mathcal{O}_\ell}(u, v) = G_{\Delta, \ell}(u, v). \]

(2.144)
Using a transformation of variables to,
\[ u = z\bar{z}, \text{ and } v = (1 - z)(1 - \bar{z}), \]
(2.145)
one can find explicit solutions of the above differential equation of the conformal block. In
general the expressions for the conformal blocks are series solutions except in special cases when
the space time dimension is even for example in \( d = 2, 4, 6 \) although the solutions become
complicated as one moves up in the dimension. We quote below the solutions of the conformal
blocks in \( d = 2, 4 \) dimensions. We will consider the generalized function of the form,
\[ k_\beta(x) = x^{\beta/2} \, _2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right), \]  
(2.146)
so that the conformal blocks can be written in terms of these functions.

1. \( d=2 \)

\[ G^{d=2}_{\Delta, \ell}(z, \bar{z}) = k_{\Delta + \ell}(z)k_{\Delta - \ell}(\bar{z}) + (z \leftrightarrow \bar{z}). \]  
(2.147)

2. \( d=4 \)

\[ G^{d=4}_{\Delta, \ell}(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left[ k_{\Delta + \ell}(z)k_{\Delta - \ell - 2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right]. \]  
(2.148)
The solution for the \( d = 6 \) case is a little complicated. From the differential equation, one can
also see the asymptotic of the conformal blocks. This is irrespective of the space time dimension
\( d \) and can be obtained just by taking the \( z, \bar{z} \to 0 \) limit. This is given by,
\[ G_{\Delta, \ell} \sim z^{\lambda_1}\bar{z}^{\lambda_2}, \text{ for } z, \bar{z} \to 0, \]
(2.149)
where the quantities \( \lambda_1 \) and \( \lambda_2 \) can be written as,
\[ \lambda_1 = \frac{\Delta + \ell}{2}, \quad \lambda_2 = \frac{\Delta - \ell}{2}. \]
(2.150)

### 2.6.1 Recursion Relations

Although the solutions of the conformal blocks for \( d \geq 6 \) are complicated, nevertheless it is
possible to extract some useful information about the forms of the conformal blocks. A useful
recursion relation relating the conformal blocks in \( d + 2 \) dimensions to those in \( d \) dimensions
will now be quoted here.

\[
\left( \frac{z - \bar{z}}{z \bar{z}} \right)^2 G^d_{\Delta, \ell} (z, \bar{z}) = G^d_{\Delta - 2, \ell + 2} (z, \bar{z}) - 4 \frac{(d + \ell + 2)(d + \ell - 1)}{(d + 2\ell - 2)(d + 2\ell)} G^d_{\Delta - 2, \ell} (z, \bar{z})
\]

\[
- 4 \frac{(d - \Delta - \ell)(d - \Delta)}{(d - 2\Delta)(d - 2\Delta + 2)} \left[ \frac{(\Delta + \ell)^2}{16(\Delta + \ell - 1)(\Delta + \ell + 1)} G^d_{\Delta, \ell} (z, \bar{z})
\right.

\[
+ \frac{(d + \ell - 2)(d + \ell - 1)^2}{4(d + 2\ell - 2)(d + 2\ell)(d + \ell - \Delta - 1)(d + \ell - \Delta + 1)} G^d_{\Delta, \ell} (z, \bar{z}) \right].
\]

(2.151)

Under certain circumstances e.g. the light-cone limit where we take \( u \ll v \ll 1 \), this recursion relation simplifies dramatically enabling us to perform calculations for a certain sector of the CFT spectrum as will be discussed in the later chapters.

### 2.7 Unitarity Bounds

In all the previous sections we have not discussed about one more ingredient i.e. **Unitarity Bounds**. We will discuss this constraint briefly in this section before moving on to the bootstrap equation. This is one final ingredient we want to emphasize on, before putting all the ingredients together and for this thesis, this is the most appropriate place for this discussion. To discuss the unitarity bounds, we will mainly follow [3], although similar discussions can be found in other sources like [14] as well. We consider two primary operators in the CFT, \( O_a \) and \( O_b \) such that the norm of the states corresponding to these operators, is normalized to unity, i.e.

\[
\langle O_a | O_b \rangle = c \delta_{ab} , \quad c = 1 .
\]

(2.152)

We will consider the norm of the descendants created from these operators \( P_\mu | O_a \rangle \) such that,

\[
(P_\mu | O_a \rangle) \dagger (P_\nu | O_b \rangle) \geq 0 .
\]

(2.153)

The norm of the descendant states is given by,

\[
(P_\mu | O_a \rangle) \dagger (P_\nu | O_b \rangle) = \langle O_a | K_\mu P_\nu | O_b \rangle = \langle O_a | [K_\mu P_\nu] | O_b \rangle ,
\]

(2.154)

since \( P_\mu \dagger = K_\mu \) and \( K_\mu | O_a \rangle = 0 \). Using the conformal algebra,

\[
[K_\mu , P_\nu] = -2iD \delta_{\mu \nu} + 2iM_{\mu \nu} .
\]

(2.155)

Since \( D | O_a \rangle = i\Delta | O_a \rangle \), we finally have,

\[
(P_\mu | O_a \rangle) \dagger (P_\nu | O_b \rangle) = 2\Delta \delta_{\mu \nu} \delta_{ab} + 2i(S_{\mu \nu})_{ab} .
\]

(2.156)
The last term can be written as,

$$2i(S_{\mu\nu})^a_b = (L^{\alpha\beta})_{\mu\nu}(S_{\alpha\beta})_b^a = L \cdot S,$$

(2.157)

where \((L^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)\) is the generator of rotations in \(SO(d)\) vector representations \((V)\). This can be written as,

$$L \cdot S = \frac{1}{2}[(L + S)^2 - L^2 - S^2].$$

(2.158)

Consider for simplicity the case \((S)\) of rank-\(\ell\) symmetric-traceless tensors. The Casimir for the rank-\(\ell\) symmetric-traceless tensors is \(\ell(\ell + d - 2)\) while for the first term the maximal contribution is \(d - 1\). Hence the maximal value of \(L \cdot S\) is,

$$L \cdot S = \frac{1}{2}((\ell - 1)(\ell - 1 + d - 2) - \ell(\ell + d - 2) - (d - 1)) = -(\ell + d - 2),$$

(2.159)

Thus the unitarity bound predicts for \(\ell > 0\),

$$\Delta \geq d - 2 + \ell.$$

(2.160)

To compute the unitarity bound for scalars, we will need to take the norm of \(P_\mu P^\mu|O\rangle\) where \(|O\rangle\) is the scalar operator which gives \(\Delta = 0\) or \(\Delta \geq (d - 2)/2\). Thus in general the unitarity bounds can be stated as,

$$\Delta \geq \begin{cases} 
\frac{d-2}{2}, & \ell = 0 \\
\ell - 2 + \ell, & \ell > 0 
\end{cases}$$

(2.161)

### 2.8 Bootstrap equation

Till now we have been discussing various properties related to the construction of the conformal blocks. In this section we will finally write down the conformal bootstrap equation. In general the bootstrap principle can be applied to various case even when the external operators are with spin. The equations stem from the following conditions:

1. **Operator content**

   Information about the operators present in the CFT spectrum. This information is contained in the *Conformal Blocks* which is the contribution of a particular conformal primary \(O_{\Delta,\ell}\) and its descendants.

2. **OPE associativity**

   This condition is stated as:
The different ways of taking the OPE in a four point function should commute. Namely,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle \sim \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}} \mathcal{G}_{\mathcal{O}},$$  \hspace{1cm} (2.162)

should be equal to

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_4(x_4)\mathcal{O}_3(x_3)\mathcal{O}_2(x_2) \rangle \sim \sum_{\mathcal{O}} C_{14\mathcal{O}} C_{23\mathcal{O}} \mathcal{G}_{\mathcal{O}},$$  \hspace{1cm} (2.163)

and so on. This relation of OPE associativity is also called crossing symmetry. This is a strong constraint and imposes severe restrictions on the conformal data needed to construct the spectrum.

3. OPE convergence

As emphasized earlier, the OPE convergence for any finite separation guarantees that we can expand any higher $n$–point ($n>3$) correlation function in various channels (12)(34), (13)(24) and (14)(23) and this convergence in all these channels can be used to put constraints on the CFT data.

The above two constraints can be stated in a more concrete manner as,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle = \langle \mathcal{O}_1(x_1)\mathcal{O}_4(x_4)\mathcal{O}_3(x_3)\mathcal{O}_2(x_2) \rangle.$$

$$\hspace{1cm} (2.164)$$

The other ways of rearrangement do not give rise to additional constraint equations. Henceforth we will consider identical scalar operators for simplicity. For identical external scalar operators, we have,

$$1 + \sum_{\Delta,\ell} P_{\Delta,\ell} G_{\Delta,\ell}(u, v) = \left( \frac{u}{v} \right)^{\Delta_{\phi}} \left( 1 + \sum_{\Delta,\ell} P_{\Delta,\ell} G_{\Delta,\ell}(v, u) \right),$$

$$\hspace{1cm} (2.165)$$

where $\Delta_{\phi}$ is the conformal dimension of the external scalar operators and $P_{\Delta,\ell}$ is the square of the OPE coefficients. The above constraint equation in (2.165) is known as the Bootstrap equation. We have considered the simple case of identical external scalars but this can also be extended for the case when the external operators have spin or the theory has some global symmetry group e.g $O(N)$. In such cases, the number of such equations will be multiple depending on the symmetry structures of the representation of operators. For this thesis, we will consider only identical external scalar operators.

2.9 Old Bootstrap: à la Polyakov

Since a part of the work will include the extension of Polyakov’s pioneering work in 1973 [15], about the application of bootstrap in the determination of the critical exponents for perturbative QFTs, we will give a brief idea about the logic before concluding this chapter. The essential idea
is to formulate the completeness condition on the intermediate operators appearing in the short distance OPE and compute its effect on the Wightmann function. With the aid of “locality” condition, this formulation can thus lead to algebraic constraint equations for the anomalous dimensions and couplings for these operators. To give the philosophy behind this work, one can start with the free scalar field theory (no internal symmetry group for now) and consider a set of local scalar operators $O_{\Delta}(x)$ with conformal dimensions $\Delta$ which becomes the operators $\phi^n$ in the free theory. Since this itself does not form a complete set, we will also need to include operators with spin $O_{\Delta,\ell}$. A further assumption is that at sufficient short distances the theory enjoys full conformal invariance (massless theory). Then due to conformal symmetry,

$$\langle O_{\Delta,\ell}(0)O_{\Delta',\ell'}(x) \rangle \propto \delta_{\Delta,\Delta'}\delta_{\ell,\ell'}.$$  \hfill (2.166)

To show that the completeness condition can imply dynamical constraint equations for the anomalous dimensions and couplings for these intermediate operators, we will consider the OPE of two scalars,

$$\phi(x)\phi(0) = \sum_{\Delta,\ell} C_{\Delta,\ell}(x)O_{\Delta,\ell}(0),$$  \hfill (2.167)

where the form of $C_{\Delta,\ell}(x)$ is fixed by conformal symmetry up to some overall constants. The dynamical conditions arise from considering,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle,$$  \hfill (2.168)

after substitution of (2.167) into the correlator and demanding that the correlator has crossing symmetry. After substitution and requiring crossing symmetry, the final step is to identify the constraint equations by setting the extra pieces inconsistent with the OPE to be zero.

The motivation for this idea is manifold. First of all, we are getting a set of constraints which are algebraic in nature as opposed to the functional constraints one gets from the conventional bootstrap approach. The constraint equations are of the form,

$$\sum_{\Delta,\ell} c_{\Delta,\ell} f(\Delta,\ell)^{(s)} + \sum_{\Delta,\ell} c_{\Delta,\ell} f(\Delta,\ell)^{(t)} + \sum_{\Delta,\ell} c_{\Delta,\ell} f(\Delta,\ell)^{(u)} = 0.$$  \hfill (2.169)

where $s$, $t$ and $u$ labels various contributions from the three channels, required to make the amplitude crossing symmetric. The algebraic equations are much easier to handle and no rigorous programming is required. Secondly, we are interested in a dimensional reduction from the 4$d$ to the strong coupling regime in 3$d$. This is facilitated by a familiar known $\epsilon$—expansion. This technique can be readily applied to the above set of dynamical constraint equations by solving order by order in $\epsilon$. As we will demonstrate in chapter 6, we were able to solve for up to $O(\epsilon^2)$ by considering the constraint equations for just scalar exchange. The $O(\epsilon^2)$ result matches with the results in the existing literature which would in general require complicated loop calculations via the approach of diagrammatic perturbation theory. One input, as will be explained in chapter 6, that we required is about the $O(\epsilon^2)$ anomalous dimension of the external...
scalar operator which was necessary for matching with the $O(\epsilon^2)$ result for the scalar exchange operators. Unfortunately this information could not be retrieved from the constraint equations à la Polyakov. For this, we resorted to the technique of [16], where the leading anomalous dimensions of the external scalar operator was derived from the consistency condition of the OPE when $\epsilon \to 0$ and the free field limit is approached. The authors of [16] relied on a particular aspect of interacting theory (specializing to $4 - \epsilon$ dimensions) which is *Multiplet Recombination* which can also be looked at as the shortening condition for conformal multiplets. The essential idea was that for interacting theories (e.g. a mass less scalar $\phi^4$ theory) with the modified equation of motion,

$$\Box \phi = \lambda \phi^3,$$  \hspace{1cm} (2.170)

where $\lambda$ is the coupling of the $\phi^4$ interaction, the multiplets $\{\phi\}$ and $\{\phi^3\}$ are primaries in the free limit ($\lambda \to 0$) but $\{\phi^3\}$ becomes the descendant of $\{\phi\}$ in the presence of interaction. Thus by considering OPE of $\phi^n \phi^{n+1} |0\rangle$ where both $\{\phi\}$ and $\{\phi^3\}$ can appear, one can see the multiplet recombination for these correlators and then taking the free limit in terms of $\epsilon \to 0$, imposes non trivial constraints on the anomalous dimensions of $\phi^n$ operators (including $n = 1$). Their finding was,

$$\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{1}{108} \epsilon^2 + O(\epsilon^3), \quad \Delta_{\phi^n \geq 2} = \frac{1}{6} n(n-1) \epsilon + O(\epsilon^2), \hspace{1cm} (2.171)$$

which was later generalized to the case of $O(N)$ vector models as well. This is one input we had to consider for our case. However, we believe that with the full set of constraint equations, this information will also be a part of the consistent solution to the constraint equations in [15]. We considered only the simplest case of scalar exchanges in our analysis (chapter 6) but the full equations involve spin $- \ell$ exchange operators as well. So it should be possible to solve the anomalous dimensions for these operators as well. Eventually we would like to see a full set of constraint equations including general operators of spin $- \ell$ that can be solved to an arbitrary order in $\epsilon$. While fetching an exact $\epsilon$ dependence of the anomalous dimension, which would mean solving the strongly coupled regime in $3d$ completely, is an ambitious goal, we are hopeful that at least numerically we will be able to see the power of these constraints.
Bibliography


Chapter 3

AdS/CFT correspondence

3.1 Introduction

Since a part of the thesis will also entail the calculations from the bulk gravity side, it is worthwhile to give a brief introduction to the Gauge-Gravity duality in general. We will provide an introduction to this subject in this chapter. The chapter is largely based on [1] and [2]. For a more elaborate discussion, one can also consult [3] and [4]. For an introduction to the correspondence from the conformal bootstrap point of view, the reader is referred to [5]. For a more string theory based motivation and approach, one can also look into [6]-[11] in the bibliography. Towards the end we will provide a brief introduction to the correlators of the stress tensors following [12, 13, 14] since a part of the thesis will also entail these calculations.

To start with, the duality in general relates a Quantum Field Theory living in $d$ space time dimensions to a classical theory of Gravity living in one higher dimension. We will consider each of these aspects below:

3.1.1 QFT

Every Quantum field theory describes a framework which unifies particle-wave duality allowing for particle creation and annihilation. This goes beyond the semi-classical expansion taking quantum effects into account in a small manner (e.g. $\alpha_{EM} \ll 1$). It also describes highly Quantum systems like QCD for which $\alpha_s \gg 1$. More generally, in the space of couplings, there are special points called fixed points and all QFTs approach one or the other fixed point in UV or IR.

Fixed points are special points in the QFT. These points are actually scale invariant in the sense that,

$$x_\mu \rightarrow \alpha x_\mu,$$

(3.1)
is a symmetry of the theory. In most cases, scale invariance actually implies conformal invariance and these Quantum Field Theories go by the special name of Conformal Field Theory. Of these, the most interesting cases are those of Gauge theories with large amount of dynamical phenomenon such as spontaneous symmetry breaking, both global and gauge, confinement, etc. A subset of these theories which are most tractable ones are the supersymmetric gauge theories e.g $\mathcal{N} = 4$ SYM. This is a maximally supersymmetric theory in four dimensions which is also exactly conformally invariant. Apart from this, there are also $\mathcal{N} = 1, 2$ SUSY theories exhibiting large amount of phenomenon as chiral symmetry breaking, confinement, etc.

### 3.1.2 Gravity

Gravity theory is a complete theory in its own right which geometrizes the gravitational force by relating it to the properties of space time, curvature, etc. It describes a variety of phenomenon like presence of singularities, black hole formation, cosmological Big-Bang and so on. A semi-classical description of gravity can also take into account the Hawking radiation, etc.

One of the interesting aspects of gravity that made possible, the advent of gauge-gravity duality is the concept of black hole entropy. For a black hole with temperature $T_{BH}$ given by,

\[ T_{BH} = \frac{\hbar}{8\pi G_N M}, \quad \text{where} \ M \ \text{is the mass of the black hole}, \tag{3.2} \]

the black hole entropy is given by,

\[ S_{BH} = \frac{A_H}{4\hbar G_N}, \quad (c = 1), \tag{3.3} \]

and $A_H$ is the area of the black hole. Following the microscopic definition of entropy given by,

\[ S = k \ln \omega, \quad \text{where} \ \omega \ \text{is the number of configurations}, \tag{3.4} \]

it was realized that the microscopic structure must be unusual since $S_{BH} \sim \text{Area}$ seems to suggest that the gravity theory must have fewer degrees of freedom and best described by a local QFT with degrees of freedom of one lower dimensional theory.

### 3.2 AdS/CFT

The missing link of the above observation is provided by the so called Anti de-Sitter space/-Conformal field theory Correspondence which says that:

"Certain Quantum field theories, in particular certain Conformal field theories living in $d$ space time dimensions can be described in terms of a quantum (in a limit $\rightarrow$ classical) description of gravity theory on $d + 1$ dimensional AdS space time"
Stated otherwise, the QFT living on the \(d\) dimensional boundary can be described by a \(d+1\) dimensional bulk gravity theory and goes by the name \textit{Bulk-Boundary Correspondence}. The radial direction of the \(AdS_{d+1}\) is the same as the energy scale in the QFT such that,

\[
UV \text{ of } QFT_d \rightarrow \text{boundary of } AdS_{d+1} \\
IR \text{ of } QFT_d \rightarrow \text{interior of } AdS_{d+1}
\]

(3.5)

For non perturbative QFTs, this correspondence relates it to a classical description of gravity on \(AdS_{d+1}\).

Although this duality is used to handle a generic strongly coupled conformal field theory living on the boundary of a classical \(AdS_{d+1}\) bulk, the most well studied example of this correspondence till date is the \(\mathcal{N} = 4\) SYM with gauge group \(SU(N)\) which is dual in the large \(N\) limit and for the large \('t Hooft coupling\) (\(\lambda\)), to type II-B supergravity theory in \(AdS_5 \times S_5\).

### 3.3 Basics of AdS space

The connection, why the underlying space time of the dual bulk gravity theory is Anti de-Sitter, is hidden in the symmetry structure of the space time. To start with, AdS is the simplest solution of the Einstein equation with a negative cosmological constant (\(\Lambda\)) given by,

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0,
\]

(3.6)

where \(R_{\mu\nu}\) is the Ricci tensor, \(R\) is the Ricci scalar and \(g_{\mu\nu}\) is the metric. AdS is a special case of the know “maximally symmetric space times” for which,

\[
R_{\alpha\beta\gamma\delta} = -\frac{1}{L^2} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}),
\]

(3.7)

where \(L\) is the AdS radius. To see the underlying symmetries, we construct the \(AdS_{d+1}\) space time by embedding in \(\mathbb{R}^{d,2}\) with the equation,

\[
-y_0^2 + \sum_{i=1}^{d} y_i^2 - y_{d+1}^2 = -L^2, \quad g_{\alpha\beta} = \text{diag}[-,+,\cdots,+,+],
\]

(3.8)

so that the metric becomes,

\[
ds^2 = -dy_0^2 + \sum_{i=1}^{d} dy_i^2 - dy_{d+1}^2.
\]

(3.9)

From this, various representations can be obtained by suitable transformations. For example:
\begin{itemize}
\item **Global AdS**
\[
y_0 = L(1 + r^2)^{\frac{1}{2}} \cos \tau, \quad y_{d+1} = L(1 + r^2)^{\frac{1}{2}} \sin \tau, \quad y_i = L r \Omega_i \]
with,
\[
r \in (0, \infty), \quad \tau \in [0, 2\pi].
\]
Further depending on a re-parametrization of either \( r = \sinh \rho \) or \( r = \tan \beta \), we get,
\[
r = \sinh \rho, \quad \tau \in (-\infty, \infty), \quad \text{Global AdS},
\]
\[
r = \tan \beta, \quad \beta \in [0, \frac{\pi}{2}], \quad \tau \in \mathbb{R}, \quad \text{Global AdS with } \mathbb{R} \times S^d.
\]
\item **Poincare AdS**
\[
y_{d+1} = \frac{L}{z} t, \quad \vec{y} = \frac{L}{z} \vec{x}, \quad y_0 - y_d = \frac{L^2}{z}, \quad y_0 + y_d = z \left[ 1 + \frac{1}{z^2} (\vec{x}^2 - t^2) \right].
\]
Induced metric has a conformally flat boundary with an overall conformal factor \( 1/z \). Note that Poincare patch is the "wedge" piece of the Global AdS space time.
\end{itemize}

Whether the AdS representation is global or local as in the Poincare patch, it has a symmetry of the \( SO(d, 2) \) group which implies that for,
\[
\Sigma \in SO(d, 2), \quad Y' = \Sigma Y \rightarrow \Sigma^\dagger g \Sigma = g.
\]

### 3.4 Large \( N \) limit

The \( \mathcal{N} = 4 \) SYM in the large \( N \) limit is the only concrete example of the boundary theory that has a dual to a gravity theory on \( AdS_5 \times S^5 \). But in general we expect the duality to hold for generic large \( N \) gauge theories on the boundary in the strongly coupled regime. To consider the validity of this statement, it is worthwhile to see what properties of the large \( N \) bit are important in obtaining the dual bulk gravity description. For example,
\[
S_{\mathcal{N}=4} = -\frac{1}{g_{YM}} \int d^4x \text{ tr } [F^2],
\]
where,
\[
F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + if_{bc}^a [A_\mu^b, A_\nu^c], \quad A_\mu^a \rightarrow A_\mu^a = U^\dagger A_\mu^a U - i U^\dagger \partial_\mu U.
\]
$f_{bc}^a$ are the structure constants of the $SU(N)$ group. The structure of the action determines the colour flow in the Feynman diagrams so that for

$$Z_N[\lambda] = \int [DA_\mu] \exp(iS_{YM}[A_\mu]) , \quad \text{with} \quad \lambda = g_{YM}^2 N \left(\text{'t Hooft coupling}\right) ,$$

one can consider the amplitudes as a function of $(\lambda, N)$ and finally arrange the Feynman diagrams as an expansion depending on their $N$ dependence. For example, for a generic Feynman diagram with $V$ vertices, $E$ internal lines and $F$ faces (closed loops with trace over an index) one finds that the $N$ dependence can be written as,

$$\left(\frac{N}{\lambda}\right)^{-E} \times \left(\frac{N}{\lambda}\right)^V \times N^F = \frac{N^{V-E+F}}{\lambda^{V-E}} .$$

Defining $V - E + F = 2 - 2h$ where $h$ is defined as the number of handles, one can finally obtain the $(N, \lambda)$ dependence of the graph as,

$$N^{2-2h} \lambda^{2h-2+F} .$$

Thus the $N$ dependence of a particular Feynman diagram is defined through the “topology” of the diagram. Of special importance are the diagrams with $h = 0$ which are also called the planar/spherical diagrams $\sim N^2$. These are the dominant contributions in the large $N$ limit with fixed $\lambda$. The perturbative expansion of a generic amplitude in $1/N$ takes the form,

$$F_N(\lambda) = \sum_{h,F} C_{h,F} N^{2-2h} \lambda^{2h-2+F} = \sum_{h=0}^\infty N^{2-2h} f_h(\lambda) , \quad \text{where} \quad f_h(\lambda) = \sum_F \lambda^{2h-2+F} C_{h,F} .$$

$f_h(\lambda)$ are all the graphs with $h$ handles. This is also called a “genus expansion” of amplitudes—reminiscent of the familiar genus expansion in String Theory. In a similar manner the correlation functions of gauge invariant operators $O_i(x)$ can be arranged as an expansion in $1/N$.

### 3.5 AdS/CFT dictionary

Precisely, the full correspondence is between,

$$\mathcal{N} = 4 \text{ SYM with } U(N) \leftrightarrow \text{II B supergravity on } AdS_5 \times S^5 .$$
The actions for the two theories are,

\[ \mathcal{L}_{N=4} = \text{tr} \left[ -\frac{1}{2g_{YM}^2} F_{\mu \nu} F_{\mu \nu} + \frac{i g}{8\pi^2} F_{\mu \nu} \tilde{F}^{\mu \nu} - i \bar{\psi}_a (\sigma^\mu D_\mu)_{a \beta} \psi^\beta - \frac{6}{2} (D_\mu \phi^M) (D^\mu \phi^M) + g_{YM} c_{ij}^M \bar{\psi}_i \left[ \phi^M, \psi_j \right] + \frac{1}{2} g_{YM}^2 \sum_{M, N} \left[ \phi^M, \phi^N \right]^2 \right] , \]

\[ S_{IIB} = \frac{1}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{g} \left[ R - \frac{\partial_\mu \tau \partial^\mu \tau}{2|\text{Im} \tau|^2} - \frac{1}{4} |F_1|^2 - |G_3|^2 - \frac{1}{2} |F_5|^2 \right] - \frac{1}{32\pi i G_N^{(10)}} \int d^{10}x A_4 \wedge \bar{G}_3 \wedge G_3 + \text{fermions} . \]

The contribution of \( F_5 \) in the solution is just a constant which we call the Cosmological constant. Thus the effective action in 5d is just the \( R + \text{constant} \) from the \( F_5 \) term. The dictionary of the correspondence is dictated by three main aspects:

1. Parameters
2. Spectrum
3. Correlators

We will discuss each of these aspects in some detail below. Note that the discussion on the parameters, spectrum and the structure of the correlators is pertaining to the duality between \( \mathcal{N} = 4 \) SYM and the II B SUGRA on \( AdS_5 \times S^5 \). We assume that the general structures and the links between the two sides of the correspondence, holds for a more general class of theories although the proof remains to be explored as of yet.

### 3.5.1 Parameters

On the boundary field theory side, the main two parameters are \( \lambda \) and \( \mathcal{N} \). \( \lambda \) determines the strength of interactions (“The quantum nature of the theory”). This is given in terms of the AdS radius as,

\[ \lambda = g_{YM}^2 \mathcal{N} \leftrightarrow \left( \frac{L_{AdS}}{l_s} \right)^4 = \lambda , \quad L_{AdS} = L_{S^5} , \]

and \( l_s \) is the string length. Thus,

\[ \lambda \gg 1 \Rightarrow L_{AdS} \gg l_s \text{ string scale} . \]

Thus string corrections can be neglected and and classical SUGRA is the leading term for small curvatures. Hence,

\[ \mathcal{L}_{\text{SUGRA}} = \frac{1}{16\pi G_N^{(10)}} \left[ R + \frac{6}{2} R^4 + \cdots \right] , \]

where string corrections proportional to \( l_s \) can be neglected. Thus we are making expansions on the gravity side about large \( \lambda \) and the corrections are proportional to \( 1/\sqrt{\lambda} \). When \( \lambda \ll 1 \),
the large $\lambda$ approximation breaks down and perturbative QFT is more useful. Of the remaining terms $\theta \leftrightarrow C_0$.

On the other hand, $1/(16\pi G_N^{(10)})$ sitting outside the gravity action plays the role of quantum expansion parameter where,

$$\frac{G_N^{(10)}}{L_{AdS}^5} = \frac{\pi^4}{2N^2} \rightarrow \frac{G_N^{(5)}}{L^5} = \frac{\pi^2}{2N^2}. \quad (3.26)$$

and the quantum loop expansion on the gravity side corresponds to the large $N$ expansion in QFT. The bottom line is that

"The classical IIB supergravity corresponds to the leading large $N$ and $\lambda \rightarrow \infty$ limit of the CFT"

### 3.5.2 Spectrum

In the gauge theory, the observables are gauge invariant operators which can be both single trace and multiple trace. Here also, we will keep in discussion in context with the canonical example of the duality but as emphasized, we expect these principles to hold for a much wider class of theories which goes beyond the known examples but are yet to be proven. For the canonical example, the single trace operators are of the kind,

$$\text{tr} \left[ \phi^M(x)\phi^N(x) \right], \quad \text{tr} \left[ \phi^{M_1}(x)\cdots\phi^{M_k}(x) \right], \quad k \ll N, \cdots. \quad (3.27)$$

The local single trace operators $O_\alpha(x)$ correspond to the fields in the gravity side (on $AdS_5$) as metric ($g_{ab}$), scalars ($\phi$) and gauge fields ($A_\mu$). Any field in the ten dimensional SUGRA can be viewed as an infinite set of fields on $AdS_5$ (Kaluza-Klein decomposition). Compactification on $S^5$ thus creates a discrete set of modes in the spectrum with only the zeroth mode surviving in the low energy effective action. For example,

$$T_{\mu\nu}(x) \quad \text{EM tensor} \leftrightarrow g_{\mu\nu}(x) \quad \text{metric},$$

$$J_{\mu}^i(x) \quad (\approx SO(6)_R) \leftrightarrow A^i_\mu(x) \quad \text{Gauge fields in } AdS_5,$$

$$L_{SYM} \leftrightarrow \phi \quad \text{dilaton},$$

$$\epsilon^{\mu\nu\rho\sigma} \text{tr} \left( F_{\mu\nu}F_{\rho\sigma} \right) \leftrightarrow C_0. \quad (3.28)$$

This correspondence between the operators in the CFT and the fields in the $5d$ gravity side is actually an equivalence of states in the Hilbert space of the two sides.

**GRAVITY SIDE**
On the gravity side, in the large $N$ limit, the fields can be viewed as weakly interacting. For example,

\[
\frac{1}{G^{(5)}_N} \int d^5 x \sqrt{g} (R - 2\Lambda) \approx \frac{1}{G^{(5)}_N} \int d^5 x \sqrt{g} \left[ (\partial h)^2 + h(\partial h)^2 + \cdots \right], \quad h \equiv \text{graviton}.
\]  

(3.29)

It is assumed that all the fields are asymptotically free and satisfies the equation of motion for free fields (for example for the scalar fields, \((\Box + m^2_\Delta)\psi^{(0)}_\Delta = 0, \quad m^2_\Delta = \Delta(\Delta - 4)\)). The two independent solutions of the equations of motion for the free fields correspond to the vev of the operators on the boundary (normalizable mode) and the couplings to external sources on the gravity side (non-normalizable mode). The precise correspondence is stated as: the non-normalizable mode of \(\psi^{(0)}_\Delta\) is associated with the corresponding boundary fields \(\bar{\psi}^{(0)}_\Delta\) by the relation,

\[
\bar{\psi}^{(0)}_\Delta(x) = \lim_{z \to 0} \psi^{(0)}_\Delta(z, \bar{x}) z^{4-\Delta},
\]

(3.30)

and given the set of fields \(\bar{\psi}^{(0)}_\Delta(x)\) on the boundary, it is assumed that a complete and unique bulk solution exists, which we call \(\psi^{(0)}_\Delta\).

**CFT SIDE**

On any \(CFT_d\), there is a mapping between operators \(\mathcal{O}_a(x)\) and states on a Hilbert space of the theory on \(\mathbb{R} \times S^{d-1}\),

\[
\mathcal{O}_a(x) \leftrightarrow |\mathcal{O}_a\rangle_{S^{d-1}}.
\]

(3.31)

The eigenvalue for the translation generator along the \(\mathbb{R}\) direction \(\tau\) is \(\Delta_a = \text{scaling dimension of } \mathcal{O}_a\) such that,

\[
\mathcal{O}_a(\lambda x) = \lambda^{-\Delta_a} \mathcal{O}_a(x),
\]

(3.32)

and,

\[
e^{-iH_\tau} |\mathcal{O}_a\rangle = e^{-i\Delta_a \tau} |\mathcal{O}_a\rangle,
\]

(3.33)

where \(H_\tau\) is the Hamiltonian corresponding to the dilatation operator in radial quantization.

The claim is that these two Hilbert spaces match on both sides. Thus

\[
\Delta_a \ (\text{in CFT}) = E_a L \ (\text{global AdS}).
\]

(3.34)

The operators on the CFT which correspond to conserved currents \((T_{\mu\nu}(x), J^{ij}_\mu(x), \cdots)\), generate global symmetry transformations in the CFT. Gauge fields define gauge invariance on the gravity side. For example,

\[
g^{ij}_{\mu\nu} \rightarrow \text{diffeomorphism invariance}, \quad A^{ij}_\mu \rightarrow SO(6) \text{ gauge invariance}.
\]

(3.35)
Global part of the diffeomorphisms on the gravity side act on the boundary as the global symmetries on the boundary theory.

### 3.5.3 Correlators

We will use the mapping \( O_a(x) \leftrightarrow \phi_a(x, z) \) to compute the correlators of the form \( \langle O_1(x_1) \cdots O_n(x_n) \rangle \) from the gravity side. To start with, consider the Euclidean \( AdS_{d+1} \) metric, \( ds^2_E = \frac{L^2}{z^2} [dt^2_E + dz^2 + d\vec{x}^2] \).

We will consider bulk scalar fields \( \{\phi_a(x, z)\} \) where \( x \in (t_E, \vec{x}) \). The partition function is given by,

\[
Z[\{\phi_a(x, z)\}] = \int \Pi_a [D\phi_a] \exp(-S_{\text{grav}}[\{\phi_a\}]) \bigg|_{\phi_a(x, z) \rightarrow \phi_0^a(x)} = Z[\{\phi_0^a(x)\}] .
\]

In the limit \( N \rightarrow \infty \) and \( G_N^5 / L^3 \rightarrow 0 \), where \( L \) is the AdS radius, \( S_{\text{grav}}[\phi_a] = S_{\text{grav}}^{cl}[\phi_a^{cl}] \) evaluated on the equation of motion for the classical solution \( \phi_a^{cl} \). The claim is that \( Z[\{\phi_0^a(x)\}] \) is the generating functional for operators \( \{O_a(x)\} \) of the boundary CFT. The generating functional for the boundary CFT is given by,

\[
Z[\{\phi_0^a(x)\}] = \int [DA][D\psi][D\phi] \exp(-S_{YM}[A, \phi, \psi] + N \int d^d x \phi_0^a(x)O_a(x) ,
\]

for which the connected piece of the correlator is of the form,

\[
\langle O_1(x_1) \cdots O_n(x_n) \rangle = \frac{1}{N^n} \frac{\delta^n \ln Z[\phi_0^a]}{\delta \phi_0^a(x_1) \cdots \delta \phi_0^a(x_n)} |_{\phi_0^a(x)=0} .
\]

For \( \lambda \gg 1 \), \( S_{\text{grav}} \) is nothing but the supergravity action on \( AdS_5 \). Further in \( N, \lambda \gg 1 \) limit with \( \lambda/N \ll 1 \), we can calculate correlators in the CFT via a classical computation with two derivative gravity on \( AdS_5 \). We will now compute the correlators for the two and three point functions from the gravity side of scalar operators and use the recipe that was described for computing in the large \( N, \lambda \) limit. For a scalar field, \( \phi(x, z) \) of mass \( m^2 \), dual to the scalar operator \( O(x) \), the quadratic part of the action is given by,

\[
S_{\phi} = \frac{c}{2} \int d^d x \ dz \ \sqrt{g} [\partial \phi]^2 + m^2 \phi^2] .
\]

Note that we consider the action \( S_{\phi} \) to be invariant under \( (x, z) \rightarrow (\lambda x, \lambda z) \). The equation of motion for \( \phi(x, z) \) is given by,

\[
(\Box_{AdS} - m^2) \phi = 0 .
\]

Using the separation of variables \( \phi(x, z) = e^{ip \cdot x} f_p(z) \), we can write the differential equation for \( f_p(z) \) as,

\[
z^2 f''_p - (d - 1)zf'_p - (z^2 p^2 + m^2 L^2)f_p = 0 ,
\]
which has two independent solutions in terms of the modified Bessel functions $I_\nu(pz)$ and $K_\nu(pz)$.

Near the boundary $z \to 0$, the form of the solution is,

$$f_p(z) \sim z^\Delta, \quad \Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}.$$  \hfill (3.43)

The solution of the classical field as $z \to 0$ is,

$$\phi_{cl}(x, z) \sim \phi_0(x)z^\Delta + \phi_1(x)z^{\Delta +}.$$  \hfill (3.44)

We take $\phi_0$ as our prescribed “source” which agrees with the prescription for $z \to 0$,

$$\phi_{cl}(x, z) \to \phi_0(x)z^{d - \Delta}, \quad \text{where } \Delta = \Delta_+.$$  \hfill (3.45)

Scale invariance of the action, $S[\phi(x, z)] = S[\phi(\lambda x, \lambda z)]$, implies that the classical scalar $\phi_{cl}(x, z)$ is invariant under scaling $(x, z) \to (\lambda x, \lambda z)$ which further implies that the scaling dimension of $\phi_0(x)$ is $\phi_0(\lambda x) = \lambda^{- (d - \Delta_+)}\phi_0(x)$. Also the operator on the boundary for which $\phi_0(x)$ acts as the source, has scaling dimensions $\Delta_+$ because of the scale invariant coupling,

$$\int d^d x \, \phi_0(x)O(x).$$  \hfill (3.46)

Unitarity in a CFT forces $\Delta$ to be real, which in turn implies that $m^2 L^2 \geq -d^2/4$. On the bulk side this guarantees that there is no instability if this limit is satisfied (Breitenlohner-Freedman bound). The arbitrariness in $\phi_1(x)$ is removed by demanding that $\phi_{cl}$ is regular everywhere in the interior of $AdS_{d+1}$ which relates $\phi_1$ to $\phi_0$. We are looking for a solution of the bulk scalar fields of the form,

$$\phi_{cl}(x, z) = \int d^d x' \, K(x - x', z)\phi(x'), \quad \text{where } \phi(x'), \text{ is the boundary value.}$$  \hfill (3.47)

$K(x - x', z)$ is called the Bulk-to-Boundary propagator and satisfies,

$$(\Box_{AdS} - m^2)K(x, x', z) = 0, \quad \text{with } K(x, x', z) \to z^{d - \Delta} \delta^d(x - x'), \quad \text{as } z \to 0,$$  \hfill (3.48)

and regular everywhere in the interior of the bulk. The solution of $K(x, x', z)$ is given by,

$$K(x, x', z) = c_{d, \Delta} \frac{z^\Delta}{[(x - x')^2 + z^2]^\Delta}. \hfill (3.49)$$

Thus for $z \to 0$,

$$\phi_{cl}(x, z) \to z^{-\Delta_-} [\phi_0(x) + O(z^2) + \cdots] + z^{\Delta_+} \left[ \int d^d x' \frac{\phi_0(x')}{|x - x'|^{2\Delta_+}} + O(z^2) + \cdots \right],$$  \hfill (3.50)

from which we can write,

$$\phi_1(x) = \int d^d x' \frac{\phi_0(x')}{|x - x'|^{2\Delta_+}}.$$

(3.51)
Plugging in the form of $\phi_{cl}$ in (3.40) and using the equations of motion for $\phi_{cl}$ we find that,

$$S^{(2)}_{cl} = \frac{c}{2} \lim_{\epsilon \to 0} \int d^d x \ \phi_{cl} \partial_z \phi_{cl} \bigg|_{z=\epsilon}. \quad (3.52)$$

In terms of $\phi_0$ and $\phi_1$ we find that,

$$S^{(2)}_{cl} = c^2 \frac{d-2\Delta_+}{N^2} \int d^d x \ \phi_0(x) \phi_0(x) + \frac{c}{2} \int d^d x \ d^d x' \ \frac{\phi_0(x) \phi_0(x')}{|x-x'|^{2\Delta_+}}. \quad (3.53)$$

Modulo the first term which is a contact term, we find that the correlator $\langle O(x)O(x') \rangle$ can be given by,

$$\langle O(x)O(x') \rangle = \frac{1}{N^2} \frac{\delta^2 S^{(2)}_{cl}}{\delta \phi_0(x) \delta \phi_0(x')} \bigg|_{\phi_0=0} = c^2 \frac{d}{N^2} \frac{1}{|x-x'|^{2\Delta_+}}, \quad (3.54)$$

which agrees with the form of the correlator coming from the boundary CFT with the definition of $c$,

$$c = \frac{1}{16\pi G^{(5)}_N} \sim N^2. \quad (3.55)$$

It should be noted that putting in the source corresponds to turning on a vev for the corresponding operator. Thus presence of the source breaks the conformal invariance by deforming the Lagrangian and introducing a scale, and thus introducing a non zero vev for the one point function $\langle O(x) \rangle$. Hence at the end of the calculations we have set the auxiliary sources $\phi_0 = 0$ to preserve the conformal invariance of the boundary theory.

We can also calculate the three and higher point functions in the bulk. As an illustration, we will take the cubic interactions of the form,

$$\frac{\tilde{g}}{3} \int d^d x \ dz \ \sqrt{g} \phi^3. \quad (3.56)$$

The typical forms of the cubic interactions are,

$$\int \frac{g_{ijk}}{3} \phi_i \phi_j \phi_k, \ \int \phi \phi h_{\mu\nu}, \ \int \phi A^\mu, \quad (3.57)$$

and so on. With the cubic interaction for the scalars, the equation of motion is modified to,

$$(\square_{AdS} - m^2) \phi = \tilde{g} \phi^2. \quad (3.58)$$

We can perform a perturbative expansion of $\phi_{cl}$ as,

$$\phi_{cl} = \phi_{cl}^{(0)} + \tilde{g} \phi_{cl}^{(1)} + \cdots, \rightarrow (\square_{AdS} - m^2) \phi_{cl}^{(1)} = (\phi_{cl}^{(0)})^2, \quad (3.59)$$

such that,

$$\phi_{cl}^{(1)}(x, z) = \int d^d x' \ dz' \ \sqrt{g} \ G(x, z, x', z')(\phi_{cl}^{(0)}(x', z'))^2, \quad (3.60)$$
where $G(x, z, x', z')$ is the bulk-to-bulk propagator satisfying,

$$ (\Box_{AdS} - m^2)G(x, z, x', z) = \frac{1}{\sqrt{g}} \delta^d(x - x') \delta(z - z'), \quad (3.61) $$

As $z \to 0$ (taking one of the bulk points to the boundary), this reduces to the usual bulk-to-boundary propagator,

$$ G(x, z, x', z') \to (z')^{\Delta} K(x, x', z). \quad (3.62) $$

Finally, we can write the action with the cubic interaction as,

$$ S_{cl}[\phi_0] = S_{cl}^{(2)}[\phi_0] + S_{cl}^{(3)}[\phi_0]. \quad (3.63) $$

Most schematically, the cubic part of the action is,

$$ S_{cl}^{(3)}[\phi_0] = \frac{\tilde{g}}{3} \int d^d x \, dz \, \sqrt{g} \left[ \prod_{i=1}^{3} K(x, x'_i, z) \phi_0(x'_i) \right]. \quad (3.64) $$

and the corresponding three point function,

$$ \langle O(x_1)O(x_2)O(x_3) \rangle = \frac{1}{N^3} \frac{\delta^d S_{cl}^{(3)}}{\delta \phi_0(x_1) \delta \phi_0(x_2) \delta \phi_0(x_3)} \bigg|_{\phi_0=0} \quad (3.65) $$

$$ = \frac{\tilde{g}}{3} \int d^d x \, dz \, \sqrt{g} \prod_{i=1}^{3} K(x, x'_i, z), $$

where each of these $K(x, x'_i, z)$ are the bulk-to-boundary propagators. As a part of the correspondence,

$$ \frac{\tilde{g}}{3} \int d^d x \, dz \, \sqrt{g} \prod_{i=1}^{3} K(x, x'_i, z) \propto C_{\Delta \ldots \Delta}(\lambda) \frac{C_{\Delta \ldots \Delta}(\lambda)}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3}|x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1}|x_1 - x_3|^{\Delta_1 + \Delta_3 - \Delta_2}. \quad (3.66) $$

The gravity calculation gives the strong coupling ($\lambda \to \infty$) limit of $C_{\Delta \ldots \Delta}(\lambda)$.

For the four point function, we can similarly keep up to quartic order terms in the interaction and the calculation will proceed along the lines of the three point functions. We will not perform the calculations but comment on the difficulties involved in the gravity calculation of the four point functions. In general the four point function is not completely fixed by conformal invariance even in the field theory side. A similar story happens in the gravity side as well, where a general $n$–pt function (where $n > 3$) from the gravity side can be written as a sum of the tree Witten diagrams. These contributions are augmented by another function called spectral function which carries the information of the exchanges in the diagrams even at the tree level. As already discussed in the previous chapter, constraints obtained from the bootstrap method in CFT is a possible way to restrict the form of the interactions and the operator content in the field theory side. We expect a similar story from the bulk side as well. Although in some cases, the explicit structure of the Witten diagrams are known, the most general case, even in...
the strong coupling regime remains yet to be fully explored even from the bulk gravity side.

### 3.6 Stress tensor two point functions: A digression

Before concluding this chapter, we would like to discuss about the two point functions of the stress tensor from the bulk gravity side. Since a part of the thesis will involve this calculation, we introduce the basic idea of the calculation in this chapter. This part will be based on the calculations in [14], [12] and [13]. In [12] and following [13], the calculation was done mainly in $AdS_5$ but it can be generalized to any dimension and for higher derivative theories in general, as done in [14]. For illustration we will consider the EH action in $AdS_5$ with the metric,

$$ds^2 = \frac{r^2}{L^2} \left( -f(r)dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{L^2}{r^2 f(r)} dr^2,$$

following [19] given by,

$$S_{EH} = \frac{1}{16\pi G_N} \int \sqrt{g} (R - 2\Lambda) + \frac{1}{8\pi G_N} \int d^4x \sqrt{h} K,$$

where the second term is the generalized Gibbons-Hawking surface term. We will consider the perturbation about the AdS vacuum,

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + h_{\mu\nu}.$$

After fixing the gauge for $h_{\mu\nu}$ one can see that we can work with just one component of the fluctuation $h_{xy}$ since the other components are related to it due to the gauge. And consider the two point function $\langle T_{xy}T_{xy} \rangle$. Considering $h_{xy} = \phi$, we find that the action, quadratic in the fluctuation $h_{xy}$ becomes,

$$S_2 = \frac{1}{16\pi G_N} \int d^5x (K_r (\partial_r \phi)^2 + K_z (\partial_z \phi)^2 + \partial_r \Gamma),$$

where $\Gamma$ is the overall surface term that is taken care off by the GH term. This action can be solved by a general ansatz of the form,

$$\phi = e^{ipz} \phi_p(r),$$

and plugging in the equation of motion for $\phi$ we find that it has the following solution,

$$\phi_p(r) = \frac{C_1}{r^2} K_2 \left( \frac{L^2 p}{r} \right) + \frac{C_2}{r^2} I_2 \left( \frac{L^2 p}{r} \right),$$

where $K_2$ and $I_2$ are the modified Bessel functions of the first and second kind. Finally plugging this back in the action (3.70), (using the equation of motion for $\phi$),

$$S_2 = \frac{1}{16\pi G_N} \int d^5x \partial_r (K_r \phi \partial_r \phi).$$
we find from the coefficient of the log $|p|$ term which amounts to neglecting all the contact terms,

$$\langle T_{xy}T_{xy}(p) = \frac{\pi^2 C_T}{320} p^4, \quad (3.74)$$

where $C_T = 40/\pi^2(L^3/16\pi G_N)$ is the central charge. To match with the result from the CFT side, we can fourier transform the above result to obtain,

$$\langle T_{xy}(x_1)T_{xy}(x_2)\rangle = \frac{C_T}{40} \frac{I_{xy,xy}}{|x_1 - x_2|^3}. \quad (3.75)$$

where $I_{ab,cd}(x,y)$ is the traceless-symmetric combination of the coordinates. Note that this is an illustrative calculation for $AdS_5$ but it essentially demonstrates that the two point function for the stress tensor is related to the central charge of the theory. As we will demonstrate later for a generic higher derivative theory of gravity in $d + 1$ bulk space time dimensions, this observation is still valid with the fact that for the higher derivative theories, the central charge receives corrections from the higher derivative couplings. Similar to the scalar theory, the three point functions of the stress tensor can also be calculated from the bulk. But in general, the calculation is tedious (even after gauge fixing). Later in this thesis, we will demonstrate an alternate way of calculating the three point functions in a particular background ($Shock\ wave\ Background$), where the three point function calculation reduces effectively to the calculation of the one point function.
Bibliography


Chapter 4

Analytical aspects of Conformal Bootstrap at large spin

4.1 Introduction

Recently it has been pointed out in the context of the AdS/CFT correspondence, that there is a connection between the CFT anomalous dimensions and the bulk Shapiro time delay [1, 2, 3, 4]. In [4] it was argued that to preserve causality, the Shapiro time delay should be positive and hence the anomalous dimensions $\gamma(n, \ell)$, of double trace operators negative. Thus it is of interest to see what happens to the anomalous dimensions $\gamma(n, \ell)$, for $n > 0$. In the literature, it has been shown using input from AdS/CFT that using the results for the four point functions of dimension-2 and dimension-3 half-BPS multiplets in $\mathcal{N} = 4$ supersymmetric SU($N$) Yang-Mills theories, to leading order in $1/N^2$, $\gamma(n, \ell) \leq 0$ for all $n$—see [5] for a recent calculation for the dimension-2 case and [6] for earlier work related to the dimension-3 case. Furthermore, in [1, 2, 3], using Eikonal approximation methods pertaining to 2-2 scattering with spin-$\ell_m$ exchange in the gravity dual, the anomalous dimensions of large-$\ell$ and large-$n$ operators have been calculated.

Analytic bootstrap methods have been used in [7, 8] to study the four point function of four identical scalar operators. It has been shown that there must exist towers of operators at large spins with twists $2\Delta_\phi + 2n$ with $\Delta_\phi$ being the conformal dimension of the scalar and $n \geq 0$ is an integer. For the case where a single tower of operator exists with twists $2\Delta_\phi + 2n$ and there is a twist gap between these operators and any other operator, one can calculate the anomalous dimensions $\gamma(n, \ell)$, of such operators. In four dimensions, the anomalous dimensions in the large spin ($\ell \gg 1$) limit for these operators for $n = 0$ are given by [7, 8, 10],

$$\gamma(0, \ell) = -\frac{c_0}{\ell^2},$$  \hspace{1cm} (4.1)
where $c_0 > 0$. This conclusion is consistent with the Nachtmann theorem [11] which predicts that the leading operators at a given $\ell$ should have twists increasing with $\ell$. However it is not known if this behaviour persists for arbitrary $n$ introduced above (for a recent study see [12]).

In this chapter we examine the anomalous dimensions $\gamma(n, \ell)$, and OPE coefficients for general CFTs following [7, 8]. Our findings are consistent with AdS/CFT predictions [1, 2, 3] where it was found that for $\ell \gg n \gg 1$, $\gamma(n, \ell) \propto -n^4/\ell^2$ while for $n \gg \ell \gg 1$, $\gamma(n, \ell) \propto -n^3/\ell$ for graviton exchange dominance in the five dimensional bulk.

**Summary of the results:**

As we will summarize below, we can calculate the anomalous dimensions and OPE coefficients for the single tower of twist $2\Delta_\phi + 2n$ operators with large spin-$\ell$ which contribute to one side of the bootstrap equation in an appropriate limit with the other side being dominated by certain minimal twist operators. In this chapter we will focus on the case where the minimal twist $\tau_m = 2$. One can consider various spins $\ell_m$ for these operators. We will present our findings for various spins separately; the case where different spins $\ell_m$ contribute together can be computed by adding up our results. We begin by summarizing the $\ell \gg n \gg 1$ case first. We note that, as was pointed out in [7], in this limit we do not need to have an explicit $1/N^2$ expansion parameter to make these claims. The $1/\ell^2$ suppression in both the anomalous dimensions and OPE coefficients does the job of a small expansion parameter.

For the dominant $\tau_m = 2, \ell_m = 0$ contribution, the anomalous dimension becomes independent of $n$ and is given by,

$$\gamma(n, \ell) = -\frac{P_m(\Delta_\phi - 1)^2}{2\ell^2},$$

(4.2)

while the correction to the OPE coefficient can be shown to approximate to,

$$C_n = \frac{1}{\hat{q}_{\Delta_\phi, n}} \partial_n (\hat{q}_{\Delta_\phi, n} \gamma_n),$$

(4.3)

in the large $n$ limit similar to the observation made in [9]. The coefficient $\hat{q}_{\Delta_\phi, n}$ is related to the MFT coefficients as shown in (4.18) later. Here $P_m$ is related to the OPE coefficient corresponding to the $\tau_m = 2, \ell_m = 0$ operator. For the dominant $\tau_m = 2 = \ell_m$ contribution, the anomalous dimension is given by,

$$\gamma(n, \ell) = \frac{\gamma_n}{\ell^2},$$

(4.4)

where,

$$\gamma_n = -\frac{15P_m^{\Delta_\phi}}{\Delta_\phi^2} [6n^4 + 12n^3(2\Delta_\phi - 3) + 6n^2(11 - 14\Delta_\phi + 5\Delta_\phi^2) + 6n(2\Delta_\phi - 3)(\Delta_\phi^2 - 2\Delta_\phi + 2) + \Delta_\phi^2(\Delta_\phi - 1)^2].$$

(4.5)

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1In [12] the dependence of $n$ in the limit $\ell \gg n \gg 1$ is extracted numerically from a recursion relation but from that approach it is not possible to make general conclusions.

2Strictly speaking we will need $\ell^2 \gg n^4$ for this to hold. Otherwise we will assume that there is a small expansion parameter.
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Figure 4.1: The variation of the anomalous dimensions $\gamma_n$ with $\Delta_\phi$ showing that some of the anomalous dimensions become positive when $\Delta_\phi < (d - 2)/2$.

Using the standard AdS/CFT normalization (see [8]), $P_m = 2\Delta_\phi^2/(45N^2)$ and hence $P_m/\Delta_\phi^2$ becomes independent of $\Delta_\phi$. Thus for $n \gg 1$, $\gamma(n, \ell)N^2 \approx -4n^4/\ell^2$, independent of $\Delta_\phi$. The coefficients $\gamma_n$ are negative for arbitrary $n$ and $\Delta_\phi \geq 1$. Interestingly (as shown in figure (4.1)) some $\gamma_n$’s can become positive if $0 < \Delta_\phi < 1$, i.e., for $\Delta_\phi$ violating the unitarity bound. To make a connection between the unitarity bound and the sign of the anomalous dimension, having the exact analytic expression above was crucial.

For general $\ell_m$ we find that the anomalous dimension behaves like

$$\gamma(n, \ell) \propto -\frac{n^{2\ell_m}}{\ell^2}, \quad (4.6)$$

for large $n$. The proportionality constant is related to the corresponding OPE coefficient. Even for this case, the anomalous dimensions are all negative for $\Delta_\phi$ respecting the unitarity bound and can be positive otherwise. Thus there appears to be an interesting correlation between CFT unitarity and bulk causality (in the sense that the sign of the anomalous dimension is correlated with the bulk Shapiro time delay [4]).

Let us make some observations. If we assume that $\ell_m \leq 2$ as in [7], our results suggest that since the $\Delta_\phi$ dependence drops out in $\gamma_n$ for $n \gg 1$, the findings are universal for any 4d CFT with a scalar of conformal dimension $\Delta_\phi$ and where in the $\ell \gg 1$ limit the spectrum is populated with a single tower of operators with twists $2\Delta_\phi + 2n$ separated by a twist gap from other operators. The explicit results given in [5, 6] are indeed consistent with the universal form of our result at large $n$. Furthermore our result is consistent with the AdS/CFT calculations in the Eikonal approximation. This gives credence to our finding that in the limit $\ell \gg n \gg 1$ the anomalous dimensions and the OPE coefficients for the $\ell_m = 2$ exchange indeed take on a universal form.
We will further extract the sub leading $1/\ell^3$ correction to the anomalous dimension for stress tensor exchange dominance and show that in the limit $\ell \gg n \gg 1$, the result is universal as well. For this we will provide a systematic way to compute the corrections to the conformal blocks starting with the differential equation.

The rest of the chapter is organized as follows: we start with the review of the analytical bootstrap methods used in [7, 8] in section (4.2). In section (4.3) we apply these methods in the limit when the spin is much larger than the twist, to cases where the $lhs$ of the bootstrap equation is dominated by either the twist-2, spin-2 operator exchange or a twist-2 scalar operator exchange. In (4.4) as a further extension, we consider the sub leading terms in the $1/\ell$ expansion and compare with known results. In section (4.5) we compare our results with the ones from AdS/CFT. Specifically we find that our results are in agreement with the results in [1, 2, 3] in both the limits. We end the chapter with a brief discussion of open questions in (4.6). Certain useful relations and formulas used for (4.2) are discussed in appendices (A.1) and (A.2). In appendix (A.3) we give a brief detail of the $n$ dependence of the coefficients $\gamma_n$ for $\ell_m > 2$ cases. In appendix (A.4) we consider the differential equations which will lead to the extensions of the large $\ell$ results in the sub leading orders in $\ell$ in (4.4). In appendix (A.5) we discuss the behavior of the corrections to the OPE coefficients $C_n$ for $\ell \gg n$ limit where we show that asymptotically (for large $n$), the coefficients $C_n$ approach the relation (4.3) while at low $n$ there are deviations. In appendix (A.6), we end with a plot showing the numerical determination of the exponent ($\beta$) of $n$ appearing in the anomalous dimension. Finally in appendix (A.7) we address the other limit where the twist is much larger than the spin. This section aims to provide an unified approach to handle both the limits ($\ell \gg n$ and $n \gg \ell$) using a saddle point analysis.

4.2 Review of the analytical approach

We begin by reviewing the key results of [7] (see also [8]) which will help us set the notation as well. Consider the scalar 4-point correlation function \( \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \). In an arbitrary conformal field theory, we have a $12 \rightarrow 34$ OPE decomposition (s-channel) given by,

\[
\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_{O} P_O g_{\tau_O, \ell_O}(u,v).
\]

Here we have used the notation $x_{ij} = |x_i - x_j|$. The variables $u$ and $v$ are the conformal cross ratios defined by,

\[
u = \frac{x_{12}^2 x_{34}^2}{x_{24}^2 x_{13}^2}, \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{24}^2 x_{13}^2}.
\]

The functions $g_{\tau_O, \ell_O}(u,v)$ are called conformal blocks or conformal partial waves [9], and they depend on the spin $\ell_O$ and twist $\tau_O$ of the operators $O$ appearing in the OPE spectrum. The twist is given by $\tau_O = \Delta_O - \ell_O$, where $\Delta_O$ is the conformal dimension of $O$. $P_O$ is a positive quantity related to the OPE coefficient. The sum goes over all the twists $\tau$ and spins $\ell$ that characterize the operators.
The 4-point function will also have a decomposition in the $14 \rightarrow 23$ channel (t-channel), and equating the two channels we will have the bootstrap equation,

$$1 + \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(u,v) = \left( \frac{u}{v} \right)^{\Delta_\phi} \left( 1 + \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u) \right). \quad (4.9)$$

We will work in the limit $u \ll v < 1$. In this limit the leading term on the lhs is the 1. However on the rhs $g_{\tau,\ell}$ has no negative power of $u$ in the small $u$ limit and all terms are vanishingly small. So we cannot reproduce the leading 1 from the rhs from a finite number of terms. In mean field theory it was shown [7] that the large $\ell$ operators produce the leading term. For a general CFT, the authors of [7] argued that in order to satisfy the leading behavior,

$$1 \approx \left( \frac{u}{v} \right)^{\Delta_\phi} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u), \quad (4.10)$$

the twists $\tau$ must have the same pattern as in MFT. To show this we have to look at the large $\ell$ and small $u$ limit of the conformal blocks,

$$g_{\tau,\ell}(v,u) = k_{2\ell}(1-z)\nu^{\tau/2}F^{(d)}(\tau,v), \quad \text{(when } |u| \ll 1 \text{ and } \ell \gg 1)$$

$$k_\beta(x) = x^{\beta/2}F_1(\beta/2,\beta/2,\beta,x). \quad (4.11)$$

Here $z$ is defined by $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$; and $F^{(d)}(\tau,v)$ is a positive and analytic function near $v = 0$ whose exact expression is not necessary for the discussion. We derive the above result later in this section. For now, we just use this to rewrite (4.10),

$$1 \approx \sum_{\tau} \lim_{z \to 0} z^{\Delta_\phi} \sum_\ell P_{\tau,\ell} k_{2\ell}(1-z) \nu^{\tau/2-\Delta_\phi}(1-v)^{\Delta_\phi}F^{(d)}(\tau,v). \quad (4.12)$$

The term in brackets are independent of $z$ and $\ell$ after taking the limit and doing the sum (over $\ell$). Then what is left is just a function of $\tau$ with a sum over $\tau$. The function $F^{(d)}(\tau,v)$ around small $v$ begins with a constant. Thus we must have $\tau/2 = \Delta_\phi$ in the spectrum. Next since $F^{(d)}(\tau,v)$ has terms with higher powers in $v$, we must have $\tau = 2\Delta_\phi + 2n$ for every integer $n$, to cancel these terms. This shows that there are operators with twists $\tau = 2\Delta_\phi + 2n$. Since these are operators in MFT, $P_{\tau,\ell} = P_{\tau,\ell}^{MFT}$ at leading order. We will now focus our attention on the sub leading terms of the bootstrap equation.

The sub leading corrections to the bootstrap equation are characterized by the anomalous dimension $\gamma(n,\ell)$ and corrected OPE coefficients $C_n$. We will assume that for each $\ell$ there is a single operator having twist $\tau \approx 2\Delta_\phi + 2n$. The bootstrap equation takes the form$^3$,

$$1 + \sum_{\ell,m} \frac{P_m}{4} u^{\tau_m/2} f_{m,\ell_m}(0,v) \approx \sum_{\tau,\ell} P_{\tau,\ell} v^{\tau/2-\Delta_\phi} u^{\Delta_\phi} f_{\tau,\ell}(v,u), \quad (4.13)$$

$^3$Our conventions for $P_m$ differ from [7] by a factor of 1/4.
which is valid up to sub leading corrections in \(u\) as \(u \to 0\). Note that the lhs demands the existence of an operator of minimal twist \(\tau_m = \Delta_m - \ell_m\) which is non-zero. We set \(u = z(1 - v) + O(z^2)\) and consider \(u \to 0\) to be \(z \to 0\). The explicit form of the function \(f_{\tau_m,\ell_m}(v)\) is given by,

\[
f_{\tau_m,\ell_m}(v) = \frac{\Gamma(\tau_m + 2\ell_m)}{\Gamma(\tau_m + \ell_m)} (1-v)^{\ell_m} \sum_{n=0}^{\infty} \frac{(\tau_m + 2\ell_m)_n}{n!} v^n \left[ 2\psi(n+1) - \psi\left(\frac{\tau_m + \ell_m + n}{2}\right) \right] - \log v. \tag{4.14}\]

Later we will set \(\tau_m = 2\) because we are particularly interested in the twist 2 primary operator or the stress tensor in the theory.

Let us now focus on the rhs where we have an infinite sum over all twists and spins. In the limit \(\ell \gg n \gg 1\) we can simplify the rhs considerably. Note that we will be working in \(d = 4\) since in the \(d = 2\) case there is no minimal twist operator with a twist gap from the identity operator \((\tau_{\text{min}}^d = 0)\). To proceed we first need to find the behavior of the conformal blocks in the above limit (in other words \(\tau_m = 2\)) and when \(|u| \ll |v| < 1\). With \(u = z(1 - v) + O(z^2)\) since \(z = (1 - v) + O(z)\), we can form a small \(z\) expansion around \(z = 0\) and then a small \(v\) expansion. To find the anomalous dimension \(\gamma(n, \ell)\) for each \(\ell\) we need to match the coefficients of the terms \(v^n \log v\) on both sides of (4.13). Considering \(\tau(n, \ell) = 2\Delta_\phi + 2n + \gamma(n, \ell)\), we can see that the log \(v\) arises from the next to the leading term in the perturbative expansion around small \(v\) given by,

\[
v^{\tau(n, \ell)/2 - \Delta_\phi} \to \frac{\gamma(n, \ell)}{2} v^n \log v. \tag{4.15}\]

The MFT coefficients take the following form in the \(\ell \gg n\) limit,

\[
P_{2\Delta_\phi + 2n, \ell}^{\ell \gg 1} \approx q_{\Delta_\phi, n} \frac{\sqrt{\pi}}{2^{2\Delta_\phi + 2n + 2\ell}\ell^{2\Delta_\phi - 3/2}}, \tag{4.16}\]

where the coefficient \(q_{\Delta_\phi, n}\) is given by,

\[
q_{\Delta_\phi, n} = \frac{8}{\Gamma(\Delta_\phi)^2} \frac{(1 - d/2 + \Delta_\phi)^2_n}{n!(1 - d + n + 2\Delta_\phi)_n}. \tag{4.17}\]

Here \((a)_n = \Gamma(a + b)/\Gamma(a)\) is the Pochhammer symbol. We will also use another notation for convenience in the later part of the chapter,

\[
\tilde{q}_{\Delta_\phi, n} = 2^{-2\Delta_\phi - 2n} q_{\Delta_\phi, n}. \tag{4.18}\]

The \(d = 4\) crossed conformal blocks are given by

\[
g_{\tau, \ell}(v, u) = \frac{(1 - z)(1 - \bar{z})}{\bar{z} - z} [k_{2\ell + r}(1 - z)k_{r-2}(1 - \bar{z}) - k_{2\ell + r}(1 - \bar{z})k_{r-2}(1 - z)], \tag{4.19}\]

where we have already defined \(k_\beta(x)\) in (4.11). As already mentioned, in the large \(\ell\) limit, the conformal blocks simplify to give (4.11). For \(\ell \gg n\) we can decompose \(k_{2\ell + r}(1 - z)\) even further
to get,
\[ k_{2\ell+r}(1-z)^{\ell+\infty} \approx \frac{2^{\tau+2\ell-1} \ell^{1/2}}{\sqrt{\pi}} K_0(2\ell \sqrt{z}). \] (4.20)

We will also need the expression for \( F^{(d)}(\tau,v) \). In \( d = 4 \) we have,
\[ F^{(4)} = \frac{2^\tau}{1-v} 2 F_{1} \left[ \frac{\tau}{2} - 1, \frac{\tau}{2} - 1, \tau - 2, v \right]. \] (4.21)

With this, the entire (log \( v \) dependent part of) rhs of (4.13) in the limit \( \ell \gg n \) can be organized into the following form,
\[ \sum_{\tau,\ell} P_{\tau,\ell} u^{\tau/2-\Delta_{\phi}} u^{\Delta_{\phi}} f_{\tau,\ell}(v,u) = \sum_{n=0,\ell=\ell_0}^{\infty} \frac{q_{\phi,n}}{2} \ell^{2\Delta_{\phi} - 2} \left[ \frac{\gamma(n,\ell)}{2} \right] v^n \log v \ell^{1/2} K_0(2\ell \sqrt{z}) z^{\Delta_{\phi}} 
(1-v)^{\Delta_{\phi}-1} 2 F_1(\Delta_{\phi} + n - 1, \Delta_{\phi} + n - 1, 2\Delta_{\phi} + 2n - 2; v). \] (4.22)

Now the overall factor of \( u^{\Delta_{\phi}} \) sitting on the rhs of (4.13) is translated into an overall factor of \( z^{\Delta_{\phi}} (1-v)^{\Delta_{\phi}} \). We assume that the anomalous dimension has the form \( \gamma(n,\ell) = \gamma_n/\ell^\alpha \). Now in the large \( \ell \) limit we can convert the sum over \( \ell \) in (4.22) into an integral given by,
\[ \int_{\ell_0}^{\infty} d\ell \ell^{-1-\alpha+2\Delta_{\phi}} z^{\Delta_{\phi}} K_0(2\ell \sqrt{z}) \approx \frac{z^{\alpha/2}}{4}\Gamma^2 \left( \Delta_{\phi} - \frac{\alpha}{2} \right) + O(z^{\Delta_{\phi}} \log z). \] (4.23)

In order to do this integral, it is convenient to use an upper cutoff \( L \). The integral works out to be in terms of regularized Hyper-geometric functions. By expanding the result assuming \( L \sqrt{z} \gg 1 \) and \( \ell_0 \sqrt{z} \ll 1 \) we get the leading and sub leading terms in the above equation. For \( \Delta_{\phi} > 1 \), the \( O(z^{\Delta_{\phi}} \log z) \) terms can be ignored. This reproduces the factor of \( z^{\alpha/2} \) exactly if \( \alpha = \tau_m \). If we take the minimal nonzero twist to be \( \tau_m = 2 \), the anomalous dimension behaves as,
\[ \gamma(n,\ell) = \frac{\gamma_m}{\ell^2}. \] (4.24)

Once again the interested reader should refer to [7, 8] for the mathematical details of the above algebra and approximations. In the next section, we demonstrate how the expression for \( \gamma_n \) can be given in terms of an exact sum for all \( n \). This sum enables us to extract the exact behavior of the anomalous dimensions for all \( n \) when \( \ell \gg n \). Later in appendix (A.7) we have also considered anomalous dimensions for the other limit \( n \gg \ell \gg 1 \).

### 4.3 The \( \ell \gg n \) case

We begin by determining \( \gamma_n \) appearing in (4.24) in the limit \( \ell \gg n \). To get \( \gamma_n \), we have to match the power of \( v^n \log v \) on both sides of (4.13). To do that we take the \( (1-v)^{\Delta_{\phi}-1} \) of (4.22) to the

\[ \text{Note that for } \Delta_{\phi} = 1 \text{ and } \tau_m = 2, \text{ this does not work as the Gamma function blows up. This is presumably indicative of a log } \ell \text{ scaling for the operators } [14] \text{ in this case.} \]
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lhs of (4.13) and expand $(1-v)^{\ell_m+\tau_m/2-\Delta\phi+1}$ in powers of $v$. Thus the lhs of (4.13) becomes,

$$-(1-v)^{\tau_m/2+\ell_m+1-\Delta\phi} P_m \frac{\Gamma(2\ell_m+\tau_m)}{4 \Gamma(\ell_m+\tau_m/2)^2} \sum_{n=0}^\infty \left( \frac{\tau_m/2+\ell_m}{n!} \right)^2 v^n \log v,$$

(4.25)

where $(a)_b$ is the Pochhammer symbol. Expanding the term $(1-v)^{\ell_m+\tau_m/2-\Delta\phi+1}$, we get,

$$(1-v)^{\ell_m+\tau_m/2-\Delta\phi+1} = \sum_{a=0}^\infty (-1)^k \frac{b!}{\alpha!(b-\alpha)!} v^\alpha \quad \text{where} \quad b = \ell_m + \frac{\tau_m}{2} + 1 - \Delta\phi. \quad (4.26)$$

Now set $n + \alpha = k$ whereby the lhs can be arranged as $\sum_{n=0}^\infty L_n v^n \log v$ where to find $L_k$ we need to perform the $\alpha$ sum explicitly.

This gives, the coefficient of $v^n \log v$ to be,

$$L_n = -4P_m \frac{\Gamma(\tau_m+2\ell_m)}{\Gamma(\frac{\tau_m}{2}+\ell_m)} \sum_{a=0}^\infty (-1)^a \left( \frac{\tau_m/2+\ell_m}{(n-a)!} \right)^2 \frac{b!}{(b-\alpha)!\alpha!}, \quad (4.27)$$

where we have multiplied the lhs of (4.13) with an overall numerical factor of 16 coming from the rhs of (4.13). This finite sum is given by,

$$L_n = \frac{4P_m \Gamma(2\ell_m+\tau_m) \Gamma(n+\ell_m+\frac{\tau_m}{2})}{\Gamma(1+n)^2 \Gamma(\ell_m+\frac{\tau_m}{2})^4} 3F_2\left(-n,-n,-1-\ell_m+\Delta\phi-\frac{\tau_m}{2}; 1-n-\ell_m-\frac{\tau_m}{2},1-n-\ell_m-\frac{\tau_m}{2},1 \right). \quad (4.28)$$

To get the same coefficient of $v^n \log v$ on the rhs of (4.13), we expand the hypergeometric function in powers of $v$ given by

$$2F_1(\tau/2-1,\tau/2-1,\tau-2,v) = \sum_{\alpha=0}^\infty \frac{(\tau/2-1)_\alpha}{(\tau-2)_\alpha \alpha!} v^\alpha,$$

(4.29)

where $(a)_b$ is the Pochhammer symbol given by $(a)_b = \Gamma(a+b)/\Gamma(a)$. On the rhs we have two infinite sums $\sum_{k=0}^\infty \sum_{\alpha=0}^\infty f_{\alpha,k} v^{k+\alpha}$. To put the rhs in the form $\sum_{n=0}^\infty R_n v^n$ we will regroup the terms in the double sum in increasing powers of $v^n$. This is achieved by setting $k+\alpha = n$ where $\alpha$ runs from 0 to $n$ giving,

$$\text{rhs} = \sum_{n=0}^\infty R_n v^n \log v,$$

(4.30)

where, the coefficients $R_k$ can be written as

$$R_n = \Gamma(\Delta\phi-\frac{\tau_m}{2})^2 \sum_{\alpha=0}^n q_{\Delta\phi,n-\alpha} \gamma_{n-\alpha} \left( \frac{(\tau/2-1)^2_{n-\alpha}}{(n-\alpha)!((\tau-2)_{n-\alpha})} \right), \quad (4.31)$$

where the extra factor of $\frac{1}{2}$ comes from the normalization $2^{2\ell+\tau-1}$ when we consider the large $\ell$ approximation of the conformal blocks. Equating the coefficients $R_n = L_n$ we can find the corresponding coefficients $\gamma_n$. Thus, in principle, we would know $\gamma_n$ if we know $\gamma_k$ for all
Figure 4.2: log $|\gamma_n|$ vs. log $n$ plot showing the dependence of $\gamma_n$ on $n$ for $n \gg 1$. $\gamma_T$ is the anomalous dimension for the spin-2 operator exchange and $\gamma_S$ for the scalar operator exchange. The slope of the blue straight line for spin-2 exchange is 3.998 while the red line denotes the scalar exchange for which $\gamma_n$ is constant for all $n$. We have used $\Delta \phi = 2$ in the above plots. The blue line data on the left vertical axis has been scaled down by a factor of $\sim$ the slope. So in essence we are plotting $1/4 \log |\gamma_T|$ vs. log $n$. Thus slope of the blue line appears to be $< 1$.

$k \leq n - 1$. In figure (4.2) we have plotted the log $\gamma_n$ vs. log $n$ for a twist-2 scalar and a twist-2 and spin-2 field.

We find that the slope of the curve for the twist-2, spin-2 exchange is $\approx 4$ while that for the twist-2 scalar is a constant. So $\gamma_n \sim n^4$ for large values of $n$ for spin-2 field. To show this behavior explicitly, we notice that $\gamma_n$ can be written as an exact sum over the coefficients $R_m$ appearing on the lhs. This formula can be guessed by looking at the first few $\gamma_n$s. We give the form of the first few $\gamma_n$s. These take the form$^5$,

$$\gamma_0 = \frac{(\Delta \phi - 1)^2}{8} L_0,$$
$$\gamma_1 = -\frac{(\Delta \phi - 1)^2}{8} L_0 + \frac{\Delta \phi - 1}{4} L_1,$$
$$\gamma_2 = \frac{(\Delta \phi - 1)^2}{8} L_0 - \frac{2\Delta \phi - 1}{4} L_1 + \frac{2\Delta \phi - 1}{2\Delta \phi} L_2 \text{ etc.} \quad (4.32)$$

We observe that the above terms follow a definite pattern which can be written as,

$$\gamma_n = \sum_{m=0}^{n} a_{n,m} \quad \text{with} \quad a_{n,m} = c_{n,m} L_m. \quad (4.33)$$

where for general $\tau_m$ and $\ell_m$ the coefficients $c_{n,m}$ are given by,

$$c_{n,m} = \frac{1}{8} \left( \frac{\Gamma(\Delta \phi)}{\Gamma(\Delta \phi + m - 1)} \right)^2 \left( \frac{(\Delta \phi + n - 3)_m}{(n - m)!} \right)^n (-1)^{n+m} \left( \frac{\Gamma(\Delta \phi - 1)}{\Gamma(\Delta \phi - \tau_m/2)} \right)^2. \quad (4.34)$$

$^5$We will assume $\Delta \phi > 1$. See footnote 4.
We have checked the analytic expression for the coefficients $\gamma_n$ agrees with the solutions of $\gamma_n$ found from solving the equations $R_k = L_k$ order by order for arbitrary values of $n$.

### 4.3.1 Case I: $\tau_m = 2$, $\ell_m = 0$

We now consider the case where the $lhs$ of (4.13) is dominated by the exchange of a twist-2 scalar operator. For this case

$$
_3F_2 \left[ \begin{array}{c}
-m, -m, -2 + \Delta \phi \\
-m, -m
\end{array} ; 1 \right] = \sum_{k=0}^{m} \frac{\Gamma(k + \Delta \phi - 2)}{\Gamma(\Delta \phi - 2)k!} = \frac{\Gamma(\Delta \phi + m - 1)}{\Gamma(m + 1)\Gamma(\Delta \phi - 1)}.
$$

(4.35)

The coefficients $a_{n,m}$ can thus be written as,

$$
a_{n,m} = -\frac{P_m (-1)^{m+n}(\Delta \phi - 1)\Gamma(n + 1)\Gamma(\Delta \phi)\Gamma(2\Delta \phi + m + n - 3)}{2 \Gamma(m + 1)\Gamma(n + 1 - m)\Gamma(\Delta \phi + m - 1)\Gamma(2\Delta \phi + n - 3)}.
$$

(4.36)

We sum over the coefficients $a_{n,m}$ to get,

$$
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{P_m}{2}(\Delta \phi - 1)^2.
$$

(4.37)

Note that the coefficients $\gamma_n$ appearing in the expression for the anomalous dimension become independent of $n$ in this case. The details can be found in appendices (A.1) and (A.2).

### 4.3.2 Case II: $\tau_m = 2$, $\ell_m = 2$

Here we consider the case where the $lhs$ of (4.13) is dominated by the exchange of a twist-2 and spin-2 operator exchange. In the language of AdS/CFT, the particle is a graviton that dominates the scattering amplitude in the Eikonal limit [1, 2, 3]. As in the previous case the anomalous dimension goes as $\sim 1/\ell^2$ for large spin in the $rhs$ of (4.13). Performing the $\ell$ integration we are left with a single sum on the $rhs$ from which we can determine the coefficients $\gamma_n$ as a function of $n$. Using relation (4.33) we can evaluate the coefficients $L_m$ for the case when $\tau_m = 2$ and $\ell_m = 2$ respectively which we proceed to show below. We defer the details of the calculation to the appendix and present here with only the final results. First we write

$$
_3F_2 \left[ \begin{array}{c}
-m, -m, -4 + \Delta \phi \\
-2 - m, -2 - m
\end{array} ; 1 \right] = \sum_{k=0}^{m} \frac{(m + 1 - k)^2(m + 2 - k)^2\Gamma(\Delta \phi - 4 + k)}{(m + 1)^2(m + 2)^2\Gamma(k + 1)\Gamma(\Delta \phi - 4)}
$$

$$
= 4(6m^2 + 6m(\Delta \phi - 1) + \Delta \phi(\Delta \phi - 1))\Gamma(m + \Delta \phi - 1)}{(m + 1)(m + 2)\Gamma(m + 3)\Gamma(\Delta \phi + 1)}.
$$

(4.38)
The combined coefficients $a_{n,m}$, after putting in the proper normalizations, can be written as,

$$a_{n,m} = (-1)^{m+n} \frac{15P_m}{\Delta_\phi} (6m^2 + 6m(\Delta_\phi - 1) + \Delta_\phi(\Delta_\phi - 1))$$

$$\times \frac{\Gamma(n+1)\Gamma(\Delta_\phi)\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n+1-m)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)}.$$

(4.39)

We can now perform the summation, over the coefficients $a_{n,m}$ to get,

$$\gamma_n = \sum_{m=0}^{\infty} a_{n,m} = -\frac{15P_m}{\Delta_\phi^2} [6n^4 + \Delta_\phi^2(\Delta_\phi - 1)^2 + 12n^3(2\Delta_\phi - 3) + 6n^2(11 - 14\Delta_\phi + 5\Delta_\phi^2)$$

$$+ 6n(2\Delta_\phi - 3)(\Delta_\phi^2 - 2\Delta_\phi + 2)].$$

(4.40)

The above formula negative and monotonic for all values of $n$ and $\Delta_\phi > 1$ (see appendices (A.1) and (A.2) for details). Until this point we did not need the explicit form of the coefficient $P_m$ but we can choose the conventions[8]. $P_m$ for any general $d$ is given by

$$P_m = \frac{d^2}{(d-1)^2} \frac{\Delta_\phi^2}{C_T}.$$ 

(4.41)

This result follows from the conformal Ward identity; as a consequence the $\Delta_\phi$ independence of the $n^4$ term in the anomalous dimension is a general result. For our case we put $d = 4$ and $C_T = 40N^2$, which correspond to the AdS$_5$/CFT$_4$ normalization and where $C_T$ is the central charge. Putting all these together, we get, $P_m = \frac{2}{45N^2} \Delta_\phi^2$. Note that the $n^4$ term in $\gamma_n$ becomes independent of $\Delta_\phi$ using this convention. Thus when $n$ is large, the result is independent of $\Delta_\phi$ and hence universal.

4.3.3 Comment on the $\mathcal{N} = 4$ result

In [5], the authors showed that for dimension-2 half-BPS multiplet the anomalous dimension in $\mathcal{N} = 4$ SYM, for $\Delta_\phi = 2$, has the form,

$$\gamma(n, \ell)N^2 = \frac{-4(n+1)(n+2)(n+3)(n+4)}{(\ell+1)(\ell+6+2n)}.$$

(4.42)

To compare this with our result (4.40) we put $P_m = 2/(45N^2)\Delta_\phi^2$ (See for eg. [8]), and set $\Delta_\phi = 4$. This gives,

$$\gamma(n, \ell)N^2 \approx \frac{-4(n+1)(n+2)(n+3)(n+4)}{\ell^2}.$$ 

(4.43)

for large values of $\ell$. Quite curiously this form matches with the supergravity result, for large spin and finite $n$. The reason for this agreement is not clear to us although [5] made a similar observation that the extra solutions to the bootstrap equation they find (for $\Delta_\phi = 2$) match exactly with the solutions in [9] for $\Delta_\phi = 4$.

---

6 We thank Joao Penedones for reminding us of this fact.
4.4 Subleading terms at large spin and large twist

So far we have concentrated on the leading $n$ dependence in the anomalous dimensions for large spin operators, when the lhs of the bootstrap equation is dominated by the stress tensor exchange. Let us now see how to derive the first sub leading term with the stress tensor exchange. In [15], this problem for leading twist was considered. By considering large twists we will extract universal results. It turns out that just keeping the leading $\ell$ dependence of the rhs is not sufficient anymore. We will assume a $1/N$ expansion so that we can use the $1/\ell$ corrections in $PMFT$—without the large $N$ we would need to keep track of corrections to these coefficients as well. In $PMFT$ the sub leading corrections at large $\ell$ take the form,

$$PMFT = \frac{\sqrt{\pi} \ell^{2\Delta_\phi - 3/2}}{2^{2\Delta_\phi + 2\ell}} \left[ q_{\Delta_\phi, n} + \frac{1}{\ell} r_{\Delta_\phi, n} \right],$$  \hspace{1cm} (4.44)

where the coefficients $q_{\Delta_\phi, n}$ and $r_{\Delta_\phi, n}$ are given by,

$$q_{\Delta_\phi, n} = \frac{8(\Delta_\phi - 1)n^2}{(2\Delta_\phi + n - 3)n \Gamma(n + 1) \Gamma(\Delta_\phi)^2},$$

$$r_{\Delta_\phi, n} = \frac{(\Delta_\phi - 1)n^2}{(2\Delta_\phi + n - 3)n \Gamma(n + 1) \Gamma(\Delta_\phi)^2} \left[ 5 - 20\Delta_\phi + 16\Delta_\phi^2 + 4n(4\Delta_\phi - 3) \right].$$  \hspace{1cm} (4.45)

The conformal blocks in the crossed channel, in the large $\ell$ and $u \to 0$ limit is,

$$g_{\tau, \ell}(v, u) \approx \int_0^1 dt (1 - t) \left[ 1 + 2(\tau - 1)z \right] K_0(2\ell \sqrt{z}) - \sqrt{z\tau} K_1(2\ell \sqrt{z}) \right].$$  \hspace{1cm} (4.49)

As we show in the appendix (A.4), for $\ell \gg n \approx 1$,

$$k_{2\ell + \tau}(1 - z) = \frac{\Gamma(2\ell + \tau)}{\Gamma(\tau/2 + \ell)^2} \int_0^1 \frac{dt}{t(1-t)} \left( \frac{(1-z)t(1-t)}{1-t(1-z)} \right)^{\ell+\tau/2}. \hspace{1cm} (4.47)$$

We can further approximate the $K_0$ function in the limit of small $\tau/\ell$ up to first order and similarly for the $\Gamma$-functions. The relevant part of the conformal block in the crossed channel takes the form,

$$k_{2\ell + \tau}(1 - z) = \frac{\Gamma(2\ell + \tau)}{\Gamma(\ell + \frac{\tau}{2})^2} K_0(2\ell \sqrt{z}) + O(z). \hspace{1cm} (4.48)$$

Upto this order is sufficient for the calculation of the first sub leading order in $z$ after $z^{m/2}$. Let us assume that the anomalous dimensions can be expanded in the form,

$$\gamma(n, \ell) = \frac{\gamma_0}{\ell^m} + \frac{\gamma_1}{\ell^{m+1}} + \frac{\gamma_2}{\ell^{m+2}} + \ldots.$$  \hspace{1cm} (4.50)
From the first sub leading correction we should be able to determine $\gamma_1^n$. We already know the coefficient $\gamma_0^n$ as an exact function of $n$ and $\Delta_\phi$. Similar to the leading order case, we can perform the large $\ell$ summation (as an integral) and then evaluate the coefficients of the sub leading powers of $z$ resulting from these extra terms. On the lhs, the sub leading powers of $z$ take the form,

$$1 + \frac{1}{4} P_m z^{\tau_m/2} g_{\tau_m, \ell_m}(u, v) = 1 + z^{\tau_m/2} f_1(v) \log v + z^{\tau_m/2 + 1} f_2(v) \log v + O(z^{\tau_m/2 + 2}). \quad (4.51)$$

Thus on the lhs only integer powers of $z$ are there in the sub leading pieces. Whereas on the rhs, after the large $\ell$ integral, the first sub leading power after the leading term in $z^{\tau_m/2}$ begins with $z^{(\tau_m+1)/2}$ with the coefficient,

$$z^{(\tau_m+1)/2} \sum_n [(1 - 2\Delta_\phi + (2n + 2\Delta_\phi)\tau_m)\gamma_0^n + 2\gamma_1^n] \frac{(\Delta_\phi - 1)_n^2 \Gamma(\Delta_\phi - \frac{1}{2} - \frac{m}{2})^2}{2\Gamma(n + 1) \Gamma(\Delta_\phi)^2 \Gamma(2\Delta_\phi + n - 3)_n} \times v^n \log v \ F^{(d)}[2\Delta_\phi + 2n, v]. \quad (4.52)$$

Since there is no $z^{(\tau_m+1)/2}$ term on the lhs, we must have for large $n$,

$$\gamma_1^n = -n\tau_m \gamma_0^n. \quad (4.53)$$

Thus specializing to the case of $\tau_m = 2$ (stress tensor), to the first sub leading order in $\ell$ we have for the anomalous dimensions,

$$\gamma(n, \ell) = \gamma^0(n, \ell) \left(1 - \frac{2n}{\ell}\right), \quad (4.54)$$

where $\gamma^0(n, \ell) = -4n^4/(N^2\ell^2)$. The sub leading correction will prove to be useful in the next section. The main result of [15] is still consistent with this finding\footnote{We thank Fernando Alday for the following observation.}. For $n = 0$ for large $\ell$ the Casimir is $j^2 \approx \ell^2$. For general $n$ the Casimir will become $j^2 \approx (\ell + n)^2$. The main conclusion of [15] is that only even powers of $j$ should appear in the large spin limit—this was explicitly shown for leading twists. In terms of $j$ (for $\tau_m = 2$) we could have $1/j^2 \sim 1/(\ell + n)^2$ or $1/(j^2 - n^2) \sim 1/((\ell + n)^2)$ so that in the $j$ variable only even powers of $j$ appear—both these forms are compatible with the sub leading term we have derived. We will find the latter behavior in what follows which is also consistent with the results of [5] for the supersymmetric $\mathcal{N} = 4$ case.

### 4.5 Comparison with AdS/CFT

AdS/CFT provides us with a formula for the anomalous dimensions in terms of the variables $\tilde{h} = \Delta_\phi + n$, $h = \tilde{h} + \ell$. In the limit $h, \tilde{h} \rightarrow \infty$, the form of the anomalous dimension is given by
where $\ell_m$ is the spin of the minimal twist operator, $\Pi(h, \bar{h})$ is a particular function of $h, \bar{h}$. In $d = 4$ the function $\Pi(h, \bar{h})$ is given by

$$
\Pi(h, \bar{h}) = \frac{1}{2\pi} \frac{h^2}{h^2 - \bar{h}^2} \left( \frac{h}{\bar{h}} \right)^{1-\Delta_m},
$$

(4.56)

where $\Delta_m$ is the dimension of the minimal twist operator. Using $\Delta_m = \tau_m + \ell_m$ for operators with minimal twist $\tau_m$ and spin $\ell_m$, the expression for the anomalous dimension in 4d becomes,

$$
\gamma_{h, \bar{h}} = -2^{2\ell_m - 3} \frac{c}{\pi} \frac{n^2 + 2\ell_m + \tau_m \ell^2}{h^2 - \bar{h}^2}.
$$

(4.57)

Neglecting the factor of $\Delta_\phi$ when both $n, \ell \gg 1$ we can write the above formula in terms $n, \ell$ giving,

$$
\gamma(n, \ell) = -2^{2\ell_m - 3} \frac{c}{\pi} \frac{n^2 + 2\ell_m + \tau_m (n + \ell)^2 - \tau_m}{\ell(2n + \ell)}.
$$

(4.58)

The functional dependence on $\ell, n$ is exactly what we found from the CFT analysis. For $\tau_m = 2$, in the limit $\ell \gg n \gg 1$ we can see that the above formula reduces to $\gamma(n, \ell) = -(2^{2\ell_m - 3} c / \pi) / (n^{2\ell_m} / \ell^2)$ while in the opposite limit it gives, $\gamma(n, \ell) = -(2^{2\ell_m - 4} c / \pi) / (n^{2\ell_m - 1} / \ell)$, where $\ell_m$ is the spin of the minimal twist operator. Further for $\ell_m = 2$, with $c = 2\pi / N^2$ our results for the two limits match exactly with the above prediction from AdS/CFT for the graviton (stress tensor) exchange. Also for $\ell_m > 2$ the $n$ and $\ell$ dependence of the above expression is the same as given by our analysis (see appendix A.3).

### 4.6 Discussions

We conclude by listing some open problems.

- It will be nice to extend our results to other dimensions, especially odd dimensions where the conformal blocks are not known in closed form.

- It will be interesting to understand if and how stringy modes can make the anomalous dimensions in the limit $n \gg \ell \gg 1$ small. We have made some preliminary observations about this limit in appendix G.

- One could use our results to develop the large spin, large twist systematics at sub leading order along the lines of [15] which considered only leading twists.

- Our results used the scalar four point function as the starting point. Whether a similar conclusion can be reached by bootstrapping other four point functions of operators with spin $\ell \neq 0$ is an interesting open problem.

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8We have recently done this in [16].
Our results agreed exactly with the large-$n$ behavior found using the Eikonal approximation in AdS/CFT. On the dual gravity side, one can try to get the sub leading terms in $n$ for the case $\ell \gg n$.

It will be very interesting to verify our claims for the $n \gg \ell \gg 1$ limit using an effective field theory approach as in [13]. In that paper it was shown how for the zero spin but large twist form of the anomalous dimension changes due to a massive mode. There it was assumed that there is no stress tensor exchange. To compare with our claims one will need to extend their analysis to arbitrary spin and allowing for a stress tensor exchange.

It will be interesting to see if Nachtmann’s original proof [11] can be extended to the $n \neq 0$ case.

As it evident from plot (4.1), that some of the anomalous dimensions may become positive when unitarity bound is violated. Can this information be used in general to infer when a theory may become non unitary.
Bibliography


Chapter 5

Universal anomalous dimensions at large spin and large twists

5.1 Introduction

For CFTs in in $d > 2$ dimensions with a scalar operator of dimension $\Delta_\phi$ and a minimal twist $(\tau = d - 2)$ stress energy tensor and using the arguments of [1, 2], one finds that there is an infinite sequence of large spin operators of twists $\tau = 2\Delta_\phi + 2n$ where $n \geq 0$ is an integer. By assuming that there is a twist gap separating these operators from other operators in the spectrum, we can go onto setting up the bootstrap equations which will enable us to extract the anomalous dimension $\gamma(n, \ell)$ of these operators [1, 2, 3]. In chapter 4, we derived $\gamma(n, \ell)$ for 4d-CFTs satisfying the above conditions in the large spin limit. We found that the $n \gg 1$ result was universal in the sense that it only depended on $n, \ell$ and $c_T$ the coefficient appearing in the two point function of stress tensors, with no dependence on $\Delta_\phi$. We were able to show an exact agreement with the Eikonal limit calculation by Cornalba et al [4, 5, 6]. With the recent resurgence in the applications [7, 8, 9, 10, 11] of the conformal bootstrap program in higher dimensions, it is of interest to ask which of these results are universal and will hold for any conformal field theory (under some minimal set of assumptions). Among other things, such results may prove useful tests for the burgeoning set of numerical tools (see e.g. [7]) being used in the program. In this chapter we will extend the analysis of chapter 4 to higher dimensions and establish the universality of the leading $n$ dependence of the anomalous dimension $\gamma(n, \ell)$ in any $d$-dimensional CFT with $d \geq 3$.

One of the key results which will enable us to perform this calculation is the derivation of a closed form expression for the conformal blocks in arbitrary dimensions in a certain approximation. This was already initiated in [1, 2] and we will take this to the logical conclusion needed to extract the anomalous dimensions. In particular, we will derive an expression that solves a recursion relation (see eq.(70) appendix A of [1] which follows from [12]) relating the blocks in
\( d \) dimensions to the blocks in \( d - 2 \) dimensions. In the large spin limit, the blocks simplify and approximately factorize. This is what allows us to perform analytic calculations.

We find that the anomalous dimensions \( \gamma(n, \ell) \) in the limit \( \ell \gg n \gg 1 \) take on the form

\[
\gamma(n, \ell) = -\frac{8(d + 1)}{c_T(d - 1)^2} \frac{\Gamma(d)^2}{\Gamma(\frac{d}{2})^{d-1}} \frac{n^d}{\ell^{d-2}}.
\] (5.1)

Here \( c_T \) is the coefficient appearing in the two point function of stress tensors. Since \( \ell \gg 1 \) we do not need \( c_T \) to be large to derive this result. However, if \( c_T \) is large we can identify the operators as double trace operators. One of the main motivations for looking into this question was an interesting observation made in [13] which relates the sign of the anomalous dimensions\(^1\) of double trace operators with Shapiro time delay suggesting an interesting link between unitarity of the boundary theory and causality in the bulk theory. We found in [3], that \( \gamma(1, \ell) \) could be positive if \( \Delta \phi \) violated the unitarity bound.

We will show that the bootstrap result is in exact agreement with the holographic calculation performed in [4]. In this calculation, one needs to assume both large spin and large twist. Moreover, why the sign of the anomalous dimension is negative as well as why the result for the anomalous dimension in this limit is independent of \( \alpha' \) corrections are somewhat obscure. We will turn to another calculation in holography, proposed in [15] which has three advantages: (a) one can consider \( \ell \gg 1 \) but \( n \) not necessarily large (b) it makes it somewhat more transparent why \( \alpha' \) corrections do not contribute to the leading order result except through \( c_T \) and (c) it relates the negative sign of the anomalous dimension to the positive sign of the AdS Schwarzschild black hole mass. We will extend the results of [15] who considered \( n = 0 \) to finite \( n \).

The chapter is organized as follows. In section 5.2, we write down a closed form expression in a certain approximation for conformal blocks \([12, 16, 17]\) in general dimensions. In section 5.3, we set up the calculation of anomalous dimensions (for double trace operators) in general dimensions \( \geq 3 \) using analytic bootstrap methods. In section 5.4, we perform the sums needed in the limit \( \ell \gg n \gg 1 \) and derive the universal result eq.(5.1). In section 5.5, we turn to holographic calculations of the same results. We conclude with a brief discussion of open problems in section 5.6. Appendix B shows the exact \( n \) dependence in \( d = 6 \) extending the \( d = 4 \) result [3].

### 5.2 Approximate conformal blocks in general \( d \)

We start with the bootstrap equation used in [1, 3] \(^2\),

\[
1 + \frac{1}{4} \sum_{\ell_m} P_n u_v^{\frac{\tau_m \ell_m}{2}} f_{\tau_m, \ell_m} (0, v) + O(u_v^{\tau_m \ell_m + 1}) = \left( \frac{u_v}{v} \right)^{\Delta \phi} \sum_{\tau, \ell} P_{\tau, \ell} g^{(d)}_{\tau, \ell} (v, u).
\] (5.2)

\(^1\)see [14] for a recent work on the sign of anomalous dimensions in \( \mathcal{N} = 4 \) Yang-Mills in perturbation theory.

\(^2\)The factor of \( \frac{1}{4} \) on the \( l.h.s \) is to match with the conventions of [3].
Chapter 5. Universal anomalous dimensions at large spin and large twists

For general $d$ dimensional CFT, the minimal twist $\tau_m = d - 2$. This is the twist for the stress tensor which we will assume to be in the spectrum. The function $f_{\tau_m,\ell_m}(0, v)$ is of the form,

$$f_{\tau_m,\ell_m}(0, v) = (1 - v)^{\ell_m} F_{\ell_m,\tau_m}(0, v)$$

(5.3)

On the rhs of (5.2), $g_{\tau,\ell}^{(d)}(v, u)$ denote the conformal blocks in the crossed channel. In the limit of the large spin, $g_{\tau,\ell}^{(d)}(v, u)$ undergo significant simplification as given in appendix B of [1]. For any general $d$, the function $g_{\tau,\ell}^{(d)}(v, u)$ can be written as,

$$g_{\tau,\ell}^{(d)}(v, u) = k_{2\ell+\tau}(1 - u) F^{(d)}(\tau, v) + O(\epsilon^{2\ell\sqrt{\tau}})$$

(5.4)

where subleading terms are exponentially suppressed at large $\ell$ and,

$$k_\beta(x) = x^{3/2} 2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right).$$

(5.5)

In the limit of large $\ell$ and fixed $\tau$ and for $u \to 0$, following appendix A of [1],

$$g_{\tau,\ell}^{(d)}(v, u) = k_{2\ell}(1 - u)^{\tau/2} F^{(d)}(\tau, v) + O(1/\sqrt{\ell}, \sqrt{u}).$$

(5.6)

Thus to the leading order we need to find the functions $F^{(d)}(\tau, v)$ to complete the derivation of the factorizaion ansatz given in (5.6).

To derive a form of the function $F^{(d)}(\tau, v)$, we start by writing down the recursion relation relating the conformal block in $d$ dimension to the ones in $d - 2$ dimensions (see [1, 12]),

$$
\left(\frac{\bar{z} - z}{(1 - z)(1 - \bar{z})}\right)^2 g_{\Delta,\ell}^{(d)}(v, u) = g_{\Delta-2,\ell+2}^{(d-2)}(v, u) - 4(\ell - 2)(d + \ell - 3)\frac{(d + \ell - 4)(d + 2\ell - 2)}{16(\Delta + \ell - 1)(\Delta + \ell + 1)} g_{\Delta,\ell+2}^{(d-2)}(v, u)
- 4(d - \Delta - 3)(d - \Delta - 2)\frac{(\Delta + \ell)^2}{(d - 2\Delta - 2)(d - 2\Delta)} g_{\Delta,\ell+2}^{(d-2)}(v, u)
- 4(d + \ell - 4)(d + \ell - \Delta - 2)^2\frac{d + \ell - \Delta - 2}{4(d + 2\ell - 4)(d + 2\ell - 2)(d + \ell - \Delta - 3)(d + \ell - \Delta - 1)} g_{\Delta,\ell+2}^{(d-2)}(v, u),
$$

(5.7)

where $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$. In the limit when $\ell \to \infty$ at fixed $\tau = \Delta - \ell$, (5.7) becomes,

$$
\left(\frac{\bar{z} - z}{(1 - z)(1 - \bar{z})}\right)^2 g_{\tau,\ell}^{(d)}(v, u) \overset{\ell \to \infty}{\sim} g_{\tau-4,\ell+2}^{(d-2)}(v, u) - g_{\tau-2,\ell}^{(d-2)}(v, u) - \frac{1}{16} g_{\tau-2,\ell+2}^{(d-2)}(v, u) + \frac{(d - \tau - 2)^2}{16(d - \tau - 3)(d - \tau - 1)} g_{\tau,\ell}^{(d-2)}(v, u) + O\left(\frac{1}{\ell}\right).
$$

(5.8)

Furthermore for $z \to 0$ and $z \ll \bar{z} = 1 - v + O(z) < 1$ we can write the lhs of (5.8) as,

$$
\left(\frac{\bar{z} - z}{(1 - z)(1 - \bar{z})}\right)^2 g_{\tau,\ell}^{(d)}(v, u) u \overset{z \to 0}{\sim} \left[\left(\frac{1 - v}{u}\right)^2 + O(u)\right] g_{\tau,\ell}^{(d)}(v, u).
$$

(5.9)
Finally putting in the factorization form in (5.6), we get,

\[(1 - v)^2 F^{(d)}(\tau, v) = 16 F^{(d-2)}(\tau - 4, v) - 2v F^{(d-2)}(\tau - 2, v) + \frac{(d - \tau - 2)^2}{16(d - \tau - 3)(d - \tau - 1)} v^2 F^{(d-2)}(\tau, v) + O(1/\sqrt{\ell}, \sqrt{u}). \] 

(5.10)

We find \(F^{(d)}(\tau, v)\) to be \(^3\),

\[F^{(d)}(\tau, v) = \frac{2^\tau}{(1 - v)^{\frac{d-2}{2}}} \sum_{k=0}^\infty \frac{\Gamma\left(\frac{\tau - d + 2}{2}\right) \Gamma\left(\frac{\tau - d + 2}{2}\right) \Gamma(-d + 2, v)}{\Gamma(1+k)\Gamma(2-d+\tau)\Gamma(1-k)} \] 

(5.11)

To see whether (5.11) satisfies the recursion relation in (5.10), we plug in (5.11) in (5.10) and expand both sides in powers of \(v\). Then the \(lhs\) is,

\[
(1 - v)^2 F^{(d+2)}(\tau, v) = \sum_{k=0}^\infty \frac{v^k}{(1-v)^{\frac{d-2}{2}}} \frac{(d^2 - 4k - 2d(-2 + \tau) + (-2 + \tau)^2) \Gamma^2 \left(-1 - \frac{d}{2} + k + \frac{\tau}{2}\right) \Gamma(-d + \tau)}{(1-v)^{\frac{d-2}{2}} \Gamma^2(1+k)\Gamma^2(-d+\tau)\Gamma(-d+k+\tau)}. 
\]

(5.12)

The \(rhs\) under power series expansion gives,

\[
16F^{(d)}(\tau - 4, v) - 2v F^{(d)}(\tau - 2, v) + \frac{(d - \tau)^2}{16(d - \tau - 1)(d - \tau + 1)} v^2 F^{(d)}(\tau, v) = \sum_{k=0}^\infty \frac{v^k}{(1-v)^{\frac{d-2}{2}}} \left( \frac{2^{-\tau+\tau}(d - \tau)^2 \Gamma^2 \left(-1 - \frac{d}{2} + k + \frac{\tau}{2}\right) \Gamma(2 - d + \tau)}{(-1 + d^2 - 2d\tau + \tau^2)\Gamma(2 - d + \tau)} \right) \right) + \frac{2^\tau \Gamma(-2 - d + \tau)\Gamma^2 \left(k - \frac{2d + \tau}{2}\right) \Gamma(-2 - d + k + \tau)}{k!\Gamma^2(-d - k + \tau)\Gamma(-1 + k)\Gamma^2(-d - k + \tau)} - \frac{2^{\tau-1} \Gamma(-d + \tau)\Gamma^2 \left(-1 + k - \frac{d - \tau}{2}\right) \Gamma(-1 - d + k + \tau)}{(-1 + k)\Gamma^2(-d - k + \tau)\Gamma(-1 - d + k + \tau)}. 
\]

(5.13)

Using properties of gamma functions the above series simplifies to the one given in (5.12). Since the two series are the same, the expression (5.11) is the solution to the recursion relation (5.10) for any \(d\). With the knowledge of \(F^{(d)}(\tau, v)\) for general \(d\), we can now do the same analysis for the anomalous dimensions in general \(d\), following what was done in [3] for \(d = 4\).

### 5.3 Anomalous dimensions for general \(d\)

\(F^{(d)}(\tau, v)\) can be written as,

\[F^{(d)}(\tau, v) = \frac{2^\tau}{(1-v)^{\frac{d-2}{2}}} \sum_{k=0}^\infty d_{\tau,k} v^k, \text{ where } d_{\tau,k} = \frac{((\tau - d)/2 + 1)k^2}{(\tau - d + 2)k!}, \] 

(5.14)

\(^3\) We guessed this form of the solution by looking at the explicit forms of \(F^{(d)}(\tau, v)\) for \(d = 2, 4\).

\(^4\) We have checked in Mathematica that the expression in (5.13) after FullSimplify matches with (5.12).

\(^5\) One can show that using (5.11) in (5.2) reproduces the exact leading term 1, on the \(lhs\), by cancelling all the subleading powers of \(v\). This shows that (5.11) is indeed the correct form of \(F^{(d)}(\tau, v)\) for general \(d\).
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where \((a)_b = \Gamma(a + b)/\Gamma(a)\). The MFT coefficients for general \(d\), after the large \(\ell\) expansion takes the form,

\[
P_{2\Delta_{\phi} + 2n, \ell} \approx \frac{\sqrt{\pi}}{4\ell} \tilde{q}_{\Delta_{\phi}, n} \ell^{2\Delta_{\phi} - 3/2}.
\]

(5.15)

where,

\[
\tilde{q}_{\Delta_{\phi}, n} = 2^{3 - \tau} \left( \frac{\Gamma(\Delta_{\phi} + n - d/2 + 1)}{n! \Gamma(\Delta_{\phi})^2 \Gamma(\Delta_{\phi} - d/2 + 1)^2} \Gamma(2\Delta_{\phi} + n - d + 1) \right).
\]

(5.16)

The rhs of (5.2) can be written as,

\[
\sum_{\tau, \ell} P^{MFT}_{\tau, \ell} v^{2\Delta_{\phi} + \Delta_{\phi}}(1 - v)^{\Delta_{\phi}} k_{2\ell}(1 - z) F^{(d)}(\tau, v).
\]

(5.17)

The twists are given by \(\tau = 2\Delta_{\phi} + 2n + \gamma(n, \ell)\). Thus in the large spin limit, to the first order in \(\gamma(n, \ell)\) we get,

\[
\sum_{n, \ell} P^{MFT}_{2\Delta_{\phi} + 2n, \ell} \left[ \frac{\gamma(n, \ell)}{2} \log v \right] v^{\ell_{2\Delta_{\phi} + \Delta_{\phi}}(1 - v)^{\Delta_{\phi}} k_{2\ell}(1 - z) F^{(d)}(2\Delta_{\phi} + 2n, v),
\]

(5.18)

where \(K_0(2\ell \sqrt{z})\) is the modified Bessel function. Moreover for large \(\ell\), the anomalous dimensions behaves with \(\ell\) like (see [1, 3]),

\[
\gamma(n, \ell) = \frac{\gamma_n}{\ell^\tau_m},
\]

(5.19)

where \(\tau_m\) is the minimal twist. With this we can convert the sum over \(\ell\) into an integral [1, 3] giving,

\[
z^{\Delta_{\phi}} \int_0^\infty \ell^{2\Delta_{\phi} - 1 - \tau_m} K_0(2\ell \sqrt{z}) = \frac{1}{4} \Gamma \left( \frac{\Delta_{\phi} - \tau_m}{2} \right)^2 z^{\tau_m/2}.
\]

(5.20)

Thus the rhs of (5.2) becomes,

\[
\frac{1}{16} \Gamma \left( \frac{\Delta_{\phi} - \tau_m}{2} \right)^2 z^{\tau_m/2} \sum_n \tilde{q}_{\Delta_{\phi}, n} \log v v^n \gamma_n (1 - v)^{\Delta_{\phi}} F^{(d)}(2\Delta_{\phi} + 2n, v).
\]

(5.21)

On the lhs of (5.2), we will determine the coefficients of \(v^\alpha\) as follows. To start with, we move the part \((1 - v)^{\Delta_{\phi} - (d - 2)/2}\) in (5.21) on the lhs to get the log \(v\) dependent part,

\[
- \frac{P_m}{4} (1 - v)^b \frac{\Gamma(\tau_m + 2\ell_m)}{\Gamma(\tau_m/2 + \ell_m)^2} \sum_{n=0}^\infty \left( \frac{(\tau_m/2 + \ell_m)n}{n!} \right)^2 v^n \log v,
\]

(5.22)

where \(b = \tau_m + \ell_m + \frac{d-2}{2} - \Delta_{\phi}\). Rearranging (5.22) as \(\sum_{\alpha=0}^\infty B_{\alpha}^d v^\alpha\), where \(n + k = \alpha\) and then summing over \(k\) from 0 to \(\infty\) gives the coefficients \(B_{\alpha}^d\),

\[
B_{\alpha}^d = -\frac{4P_m \Gamma(\tau_m + 2\ell_m)}{\Gamma(\tau_m/2 + \ell_m)^4} \sum_{k=0}^\alpha (-1)^k \frac{(\tau_m/2 + \ell_m)_{\alpha-k}}{(\alpha-k)!} \frac{b!}{k!(b-k)!}.
\]

(5.23)

\[
= -\frac{4P_m \Gamma(\tau_m + 2\ell_m)\Gamma(\tau_m/2 + \ell_m + \alpha)^2}{\Gamma(1 + \alpha)^2 \Gamma(\tau_m/2 + \ell_m)^4} \\
\times \binom{\alpha}{\ell_m - \frac{\tau_m}{2} - \alpha, 1 - \ell_m - \frac{\tau_m}{2} - \alpha, 1}.\]
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The coefficient of $v^\alpha \log v$ in (5.21) thus becomes,

$$H_\alpha = \Gamma \left( \Delta_\phi - \frac{\tau_m}{2} \right)^2 \frac{\alpha}{2} \sum_{k=0}^{\alpha} 2^{2\Delta_\phi + 2\alpha - 2k} q^{\Delta_\phi, \alpha - k} d_{2\Delta_\phi + 2\alpha - 2k, k} \gamma_{\alpha - k},$$

where we have put $\tau = 2\Delta_\phi + 2n$ and further $n = \alpha - k$ due to the fact that we have regrouped terms in $v$ with same exponent $\alpha$ and $d_{p,k}$ is given by 5.14. Equating $H_\alpha = B^d_\alpha$ for general $d$ dimensions\(^6\), we get following the steps in [3],

$$\gamma_n = \sum_{m=0}^{n} c^d_{n,m} \beta^d_m,$$

where for general $d$ dimensions, the coefficients $c^d_{n,m}$ takes the form,

$$c^d_{n,m} = \frac{(-1)^{m+n}}{8} \frac{\Gamma(\Delta_\phi)^2}{(\Delta_\phi - d/2 + 1)^m} \frac{n!}{(n-m)!} \frac{(2\Delta_\phi + n + 1 - d)_m}{\Gamma(\Delta_\phi - \tau_m/2)^2},$$

where $\tau_m = d - 2$ for non zero $\ell_m$ and the general $d$ dimensional coefficients $B^d_m$ for general $d$ dimensions is given in (5.23).

### 5.4 Leading $n$ dependence of $\gamma_n$

To find the full solution\(^7\) for the coefficients $\gamma_n$ of the anomalous dimensions in general dimensions is a bit tedious. Nonetheless we can always extract the leading $n$ dependence of the coefficients for large $n$. $\gamma_n$ can be written as,

$$\gamma_n = \sum_{m=0}^{n} a_{n,m},$$

where,

$$a_{n,m} = \frac{P_m (-1)^{m+n} \Gamma(n+1) \Gamma(2+2\delta) \Gamma(\Delta_\phi)^2 \Gamma(1+\delta+m)^2 \Gamma(1+m+n-2\delta+2\Delta_\phi)}{2\Gamma(1+\delta)^4 \Gamma(1+n-2\delta+2\Delta_\phi) \Gamma(n-m+1) \Gamma(m+1)^2 \Gamma(1+m-\delta+\Delta_\phi)^2} \times \sum_{k=0}^{m} \frac{(-m)_k^2 (x)_k}{(-m-\delta)_k^2 k!},$$

where the summand is obtained by writing out $3F_2$ in (5.23) explicitly, $\delta = d/2$ and $x = \Delta_\phi - 2\delta$. To find the leading $n$ dependence we need to extract the leading $m$ dependence inside the summation in (5.28). To do that we expand the summand around large $m$ upto $2\delta$ terms and then perform the sum over $k$ from 0 to $m$. The large $m$ expansion takes the form,

$$\sum_{k=0}^{m} \frac{(-m)_k^2 (x)_k}{(-m-\delta)_k^2 k!} m^{\geq 1} \approx \frac{\Gamma(2\delta+1) \Gamma(m+x+1)}{\Gamma(m+1) \Gamma(x+2\delta+1)} + \cdots.$$  

\(^6\)In [18], there was a numerical study of this recursion relation for $d = 4$.  
\(^7\)We present the $d = 6$ solution in the appendix. The $d = 4$ solution can be found in [3].
The terms in \( \cdots \) are the subleading terms which will not affect\(^8\) the leading order result in \( n \). Thus to the leading order,

\[
a_{n,m} \approx - \frac{P_n(-1)^{m+n} \Gamma(n+1) \Gamma(2+2\delta) \Gamma(\Delta_\phi)^2 \Gamma(1+\delta+m)^2 \Gamma(1+m+n-2\delta+2\Delta_\phi)}{2 \Gamma(1+\delta)^4 \Gamma(1+n-2\delta+2\Delta_\phi) \Gamma(n-m+1) \Gamma(m+1)^2 \Gamma(1+m-\delta+\Delta_\phi)^2} \\
\times \frac{\Gamma(2\delta+1) \Gamma(m+\Delta_\phi-2\delta+1)}{\Gamma(m+1) \Gamma(\Delta_\phi+1)}.
\]

(5.30)

We can now separate the parts of \( a_{n,m} \) into,

\[
a_{n,m} = - \frac{P_n(-1)^{m+n} \Gamma(2\delta+1) \Gamma(2+2\delta) \Gamma(\Delta_\phi)^2}{2 \Gamma(1+\delta)^4 \Gamma(1+n-2\delta+2\Delta_\phi) \Gamma(\Delta_\phi+1)} \times \frac{\Gamma(1+m+n-2\delta+2\Delta_\phi)}{\Gamma(1+m-\delta+\Delta_\phi)} \\
\times \frac{n!}{m!(n-m)!} \left[ \frac{\Gamma(m+\delta+1)^2 \Gamma(m+\Delta_\phi-2\delta+1)}{\Gamma(m+1)^2 \Gamma(m+\Delta_\phi-\delta+1)} \right].
\]

(5.31)

The leading term inside the bracket is \( m^\delta \). Thus to the leading order in \( m \), the coefficient \( a_{n,m} \) is given by,

\[
a_{n,m} = - \frac{P_n(-1)^{m+n} \Gamma(2\delta+1) \Gamma(2+2\delta) \Gamma(\Delta_\phi)^2}{2 \Gamma(1+\delta)^4 \Gamma(1+n-2\delta+2\Delta_\phi) \Gamma(\Delta_\phi+1)} \times \frac{\Gamma(1+m+n-2\delta+2\Delta_\phi)}{\Gamma(1+m-\delta+\Delta_\phi)} \\
\times \frac{n! m^\delta}{m!(n-m)!},
\]

(5.32)

for general \( d \) dimensions. Note that to reach (5.32), we just extracted the leading \( m \) dependence without making any assumptions about \( \delta \) apart from \( \delta > 0 \) and \( \delta \in \mathbb{R} \). More specifically \( \delta \) takes integer values for even dimensions and half integer values for odd dimensions. We will now need to analyze the \( m \) dependent part of \( a_{n,m} \). For that note first that using the reflection formula,

\[
\Gamma(1+m-\delta+\Delta_\phi) \Gamma(\delta-m-\Delta_\phi) = (-1)^m \frac{\pi}{\sin(1-\delta+\Delta_\phi)\pi},
\]

(5.33)

we can rewrite the coefficients \( a_{n,m} \) as,

\[
a_{n,m} = (-1)^{n+1} \frac{\sin(\Delta_\phi - \delta) \pi}{\pi} P_n n! \frac{\Gamma(2\delta+1) \Gamma(2+2\delta) \Gamma(\Delta_\phi)^2}{2 \Gamma(1+\delta)^4 \Gamma(\Delta_\phi+1) \Gamma(1+n-2\delta+2\Delta_\phi)} \\
\times \frac{m^\delta}{m!(n-m)!} \Gamma(1+m+n-2\delta+2\Delta_\phi) \Gamma(\delta-m-\Delta_\phi).
\]

(5.34)

Further using the integral representation of the product of the \( \Gamma \)-functions,

\[
\Gamma(1+m+n-2\delta+2\Delta_\phi) \Gamma(\delta-m-\Delta_\phi) = \int_0^\infty \int_0^\infty dx \, dy \, e^{-(x+y)^m+n-2\delta+2\Delta_\phi y^\delta-m-\Delta_\phi-1},
\]

(5.35)

---

\(^8\) This can be explicitly checked in Mathematica.
we pull the $m$ dependent part of $a_{n,m}$ inside the integral and perform the summation over $m$ to get,

$$
\gamma_n = (-1)^{n+1} \frac{\sin(\Delta \phi - \delta) \pi}{\pi} \frac{P_m n! \Gamma(2\delta + 1) \Gamma(2 + 2\delta) \Gamma(\Delta \phi)^2}{2\Gamma(1 + \delta)^4 \Gamma(\Delta \phi + 1) \Gamma(1 + n - 2\delta + 2\Delta \phi) \Gamma(n - \delta)} \times \int_0^\infty \int_0^\infty dx dy \ e^{-(x+y)} x^{n-2\delta+2\Delta \phi} y^{\delta - \Delta \phi - 1} \sum_{m=0}^n \left( \frac{x}{y} \right)^m \frac{m^\delta}{m!(n-m)!}.
$$

(5.36)

At this stage we will need the explicit information about whether $\delta$ is integer or half integer.

### 5.4.1 Even $d$

For even dimensions, $\delta$ is an integer. The summation over $m$ can be performed with ease now. We first put $z = \log(x/y)$. After this substitution, the summand can be written as,

$$
\sum_{m=0}^n \left( \frac{x}{y} \right)^m \frac{m^\delta}{m!(n-m)!} = \sum_{m=0}^n \frac{m^\delta}{m!(n-m)!} e^{mz}.
$$

(5.37)

It is easy to see that starting from a function $f_n(z)$ and acting $\partial_z$ repeatedly on it can give back the above sum provided,

$$
f_n(z) = \sum_{m=0}^n \frac{e^{mz}}{m!(n-m)!} = \frac{(1+e^z)^n}{n!}.
$$

(5.38)

For integer $\delta$,

$$
\sum_{m=0}^n \frac{m^\delta}{m!(n-m)!} e^{mz} = \partial_z^\delta f_n(z).
$$

(5.39)

The leading order term in the derivatives of the generating function $f_n(z)$ is of the form

$$
\partial_z^\delta f_n(z) = \frac{\Gamma(n+1)}{n! \Gamma(n+1-\delta)}(1+e^z)^n(1+e^{-z})^{-\delta} + \cdots,
$$

(5.40)

where $\cdots$ represent the subleading terms in $n$. Substituting for $z$ and then into the integral representation in (5.36), we get,

$$
\gamma_n = (-1)^{n+1} \frac{\sin(\Delta \phi - \delta) \pi}{\pi} \frac{P_m n! \Gamma(2\delta + 1) \Gamma(2 + 2\delta) \Gamma(\Delta \phi)^2}{2\Gamma(1 + \delta)^4 \Delta \phi \Gamma(1 + n - 2\delta + 2\Delta \phi) \Gamma(n - \delta)} \times \int_0^\infty \int_0^\infty dx dy \ e^{-(x+y)} x^{n-2\delta+2\Delta \phi} y^{\delta - \Delta \phi - 1} \left( \frac{x+y}{y} \right)^n \left( \frac{x}{x+y} \right)^\delta.
$$

(5.41)

The integral can be performed by a substitution of variables $x = r^2 \cos^2 \theta$ and $y = r^2 \sin^2 \theta$ and integrating over $r = 0, \infty$ and $\theta = 0, \frac{\pi}{2}$. This gives,
\[\gamma_n = -\frac{\sin(\Delta_\phi - \delta)\pi}{\pi} \frac{P_m n! \Gamma(2\delta + 1)\Gamma(2 + 2\delta)\Gamma(\Delta_\phi)}{2\Gamma(1 + \delta)^4\Delta_\phi^2 \Gamma(n + 1)\Gamma(1 + n - \delta + 2\Delta_\phi)\Gamma(n - \delta)!^2 \csc(\Delta_\phi - \delta)\pi} \frac{\Gamma(1 + n - \delta + 2\Delta_\phi)}{\Gamma(1 + \Delta_\phi)}.\] (5.42)

The leading term in \(n\) for the last ratio in (5.42) is \(n^{2\delta}\) giving the leading \(n\) dependence of \(\gamma_n\) for even dimensions as,

\[\gamma_n = -\frac{P_m \Gamma(2\delta + 1)\Gamma(2 + 2\delta)}{2\Gamma(1 + \delta)^4\Delta_\phi^2} n^{2\delta}.\] (5.43)

### 5.4.2 Odd \(d\)

In odd \(d\) as well we will start with (5.36) but in this case the calculation is slightly different from the previous one in the sense that now \(\delta\) takes half integer values. We start by writing \(\delta = p - 1/2\) where \(p \in \mathbb{Z}\), and separate the \(\sqrt{m}\) contribution by an integral representation,

\[\sqrt{\pi} \sqrt{m} = \int_{-\infty}^{\infty} e^{-mt^2} dt.\] (5.44)

We can now write the sum in (5.36) for half integer \(\delta\) as,

\[\sum_{m=0}^{n} \frac{m^p e^{mz}}{\sqrt{mm!}(n-m)!} = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{n} \frac{m^p}{m! (n-m)!} \int_{-\infty}^{\infty} e^{m(z-t^2)} dt.\] (5.45)

Here \(p \in \mathbb{Z}\). The generating function \(f_n(z)\) for this case which provides with the above expression is,

\[f_n(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(1 + e^{z-t^2})^n}{n!} dt,\] (5.46)

and (5.45) is,

\[\sum_{m=0}^{n} \frac{m^p e^{mz}}{\sqrt{mm!}(n-m)!} = \partial_z^p f_n(z).\] (5.47)

The leading order term in \(\partial_z^p f_n(z)\) is given by,

\[\partial_z^p f_n(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1)}{n!\Gamma(n+1-p)} \int_{-\infty}^{\infty} dt \left\{ (1 + e^{z-t^2})^{n-p}(e^{z-t^2})^p + \cdots \right\},\] (5.48)

where \(\cdots\) are the subleading terms which does not affect the leading order results. We can now determine the integral by saddle point method by letting the integrand as,

\[e^{g(t,z)}, \text{ where } g(t, z) = (n-p)\log[1 + e^{z-t^2}] + p(z-t^2).\] (5.49)
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Setting $g'(t,z) = 0$ we see that the saddle is located at $t = 0$. Expanding around the saddle point up to second order the integrand becomes,

$$\exp[g(t,z)] = \exp[g(0,z) + \frac{1}{2} g''(0,z) t^2 + \cdots],$$

(5.50)

where the terms in $\cdots$ contribute at $O(1/n)$ after carrying out the integral over $t$. Thus to the leading order,

$$\partial_p z f(n)(z) = 1\sqrt{\pi n!\Gamma(n+1-p)} \sqrt{-g''(0,z)} e^{g(0,z)} = \frac{\Gamma(n+1)}{n!\Gamma(n+1-p)} \frac{1}{\sqrt{n}} (1+e^z)^n(1+e^{-z})^{-\delta},$$

(5.51)

where $\delta = p - 1/2$. Substituting for $z$, we get the leading order in $n$ as,

$$\partial_p z f(n) = n^{\delta} \left( \frac{x+y}{y} \right)^n \left( \frac{x}{x+y} \right)^\delta.$$  

(5.52)

We can now substitute (5.52) into (5.36) and carry out the integral over $x$ and $y$ giving us back,

$$\gamma_n = -\frac{P_m \Gamma(2\delta+1)\Gamma(2\delta+2)}{2\Gamma(1+\delta)^4 \Delta^2_\phi} \frac{\Gamma(1+n-\delta+2\Delta_\phi)}{\Gamma(1+n-2\delta+2\Delta_\phi)} n^{\delta}.$$  

(5.53)

The large $n$ limit of the last ratio of the $\Gamma$-functions is again $n^\delta$ giving,

$$\gamma_n = -\frac{P_m \Gamma(2\delta+1)\Gamma(2\delta+2)}{2\Gamma(1+\delta)^4 \Delta^2_\phi} n^{2\delta}.$$  

(5.54)

Thus for both even and odd dimensions we get the universal result for the leading $n$ dependence for the coefficients $\gamma_n$,

$$\gamma_n = -\frac{P_m \Gamma(d+1)\Gamma(d+2)}{2\Gamma(1+d/2)^4 \Delta^2_\phi} n^{d}.$$  

(5.55)

Using $P_m = \frac{d^2 \Delta^2_\phi}{(d-1)^2 c_T}$ [2],

$$\gamma_n = -\frac{d^2 \Gamma(d+1)\Gamma(d+2)}{2(d-1)^2 \Gamma(1+d/2)^4 c_T} n^{d} = -\frac{8(d+1)}{c_T(d-1)^2} \frac{\Gamma(d)^2}{\Gamma(d/2)^4} n^{d}.$$  

(5.56)

In fig.(5.1) we show plots for $\gamma_n$ for various dimensions for different $\Delta_\phi$. At large $n$ the plots coincide which proves the universality of the leading order $n$ dependence of $\gamma_n$.

5.5 CFT vs. Holography

This section will focus on the matching of the findings from CFT with predictions from holography. We start by matching the CFT coefficient $P_m$ with the Newton constant $G_N$ appearing in the holographic calculations.
5.5.1 Comparison with the Eikonal limit calculation

Following [4, 5, 6] we find that in terms of the variables \( h = \Delta_\phi + \ell + n \) and \( \bar{h} = \Delta_\phi + n \), the anomalous dimension \( \gamma_{h,\bar{h}} \) is,

\[
\gamma_{h,\bar{h}} = -16G_N(h\bar{h})^{j-1}\Pi_\perp(h,\bar{h}) ,
\]

where \( \Pi(h,\bar{h}) \) is the graviton \((j = 2)\) propagator given by,

\[
\Pi_\perp(h,\bar{h}) = \frac{1}{2\pi^{d/2-1}} \frac{\Gamma(\Delta - 1)}{\Gamma(\Delta - d/2 + 1)} \left( \frac{h - \bar{h}}{hh} \right)^{1-\Delta} 2F_1 \left[ \Delta - 1, \frac{2\Delta - d + 1}{2}, 2\Delta - d + 1; -\frac{4h\bar{h}}{(h-h)^2} \right],
\]

and \( \Delta = d \) for the graviton. Thus the overall factors multiplying \( \gamma_{h,\bar{h}} \) are given by,

\[
\gamma_{h,\bar{h}} = -\frac{8G_N}{\pi^{d/2-1}} \frac{\Gamma(d-1)}{\Gamma(d/2+1)} (h\bar{h})g(h,\bar{h}) ,
\]

where \( g(h,\bar{h}) \) is the remaining function of \( h \) and \( \bar{h} \). In the limit \( h,\bar{h} \to \infty \) and \( h \gg \bar{h} \),

\[
\gamma_{h,\bar{h}} = -\frac{8G_N}{\pi^{d/2-1}} \frac{\Gamma(d-1)}{\Gamma(d/2+1)} \frac{\bar{h}^d}{(h-h)^{d-2}} = -\frac{8G_N}{\pi^{d/2-1}} \frac{\Gamma(d-1)}{\Gamma(d/2+1)} \frac{n^d}{\ell^{\tau_m}},
\]

where \( \tau_m = d - 2 \) for the graviton. Following [2], \( G_N \) can be related to the central charge \( c_T \) as,

\[
G_N = \frac{d+1}{d-1} \frac{1}{2\pi c_T} \frac{\Gamma(d+1)\pi^{d/2}}{\Gamma(d/2)^3} ,
\]
which leads to $\gamma_{\hat{h},\bar{h}} = -\gamma_n/\ell^m$ that matches exactly with the $\gamma_n$ from (5.56). Note that we needed $\ell \gg n \gg 1$ in the holographic calculation.

5.5.2 Another gravity calculation

In this section we will extend the calculation in [15] to non-zero $n$. Unlike the eikonal method this approach has the advantage of considering $n = 0$ as well. We leave the exact (in $n$) matching as an important and interesting open problem and address the leading order ($n \gg 1$) calculation in the present chapter. We list the salient features of the calculation below.

- The key idea is to write down a generic double trace operator $[O_1O_2]_{n,\ell}$ of quantum number $(n,\ell)$ formed from the descendants of operators $O_1$ and $O_2$ in the field theory, having quantum numbers $(n_1,\ell_1)$ and $(n_2,\ell_2)$ where

$$\ell_1 + \ell_2 = \ell, \text{ and } n_1 + n_2 = n.$$  \hfill (5.62)

- Such an operator is given by,

$$[O_1O_2]_{n,\ell} = \sum_{i,j} c_{ij} \partial_{\mu_1} \cdots \partial_{\mu_{\ell_1}} (\partial^2)^{n_1} O_1 \partial_{\mu_1} \cdots \partial_{\mu_{\ell_2}} (\partial^2)^{n_2} O_2,$$  \hfill (5.63)

where $c_{ij}$ are the Wigner coefficients that depend on $n_1, n_2, \ell_1$ and $\ell_2$. For the $n = 0$ case and for large $\ell$, $\ell_1 = \ell_2 = \ell/2$. For the $n \neq 0$ it is reasonable to assume that the maxima of $c_{ij}$ will occur, in addition to the $\ell$ variable, for $n_1 = n_2 = n/2$. Thus $\partial^2$ will be equally distributed between the two descendents $O_1$ and $O_2$.

- From holography we can model the two descendents $O_1$ and $O_2$ as two uncharged scalar fields with a large relative motion with respect to each other (corresponding to $\ell \gg 1$) or equivalently one very massive object which behaves as an AdS-Schwarzschild\(^9\) black hole and the other object moves around it with a large angular momentum proportional to $\ell$.

- The calculation of the anomalous dimension for $[O_1O_2]_{n,\ell}$ from the field theory is then equivalent to the holographic computation of the first order shift in energy of the above system of the object rotating around the black hole.

The AdS-Schwarzschild black hole solution in $d + 1$ dimensional bulk is given by,

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)} dr^2 + r^2 d\Omega^2,$$  \hfill (5.64)

where,

$$U(r) = 1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{R_{AdS}^2}.$$  \hfill (5.65)

\(^9\)A cleaner argument may be to replace the AdS-Schwarzschild black hole with the AdS-Kerr black hole.
The mass of the black hole is given by,

$$M = \frac{(d - 1)\Omega_{d-1}\mu}{16\pi G_N}. \quad (5.66)$$

The first order shift in the energy is given by\textsuperscript{10},

$$\delta E_{\text{orb}}^d = \langle n, \ell_{\text{orb}}|\delta H|n, \ell_{\text{orb}}\rangle = -\frac{\mu}{2} \int r^{d-1}dr d^{d-1}\Omega \langle n, \ell_{\text{orb}}| \left( \frac{r^{2-d}}{1+r^2} \right)^2 (\partial_t \phi)^2 + r^{2-d}(\partial_r \phi)^2 \rangle |n, \ell_{\text{orb}}\rangle. \quad (5.67)$$

The label ‘\text{orb}’ implies that currently we are considering one of the masses of the binary system. We can also add higher derivative corrections coming from the $\alpha'$ corrections to the metric. This is one of the advantages of doing the anomalous dimension calculation in this approach. It will make transparent the fact that the $\alpha'$ corrections will not affect the leading result. The metric will be modified\textsuperscript{11} by adding corrections to the factor $r^{2-d}(1 + c_h\alpha' h r^{-2h})$ where $h$ is the order of correction in $\alpha'$. The wavefunction of the descendant state derived from the primary is given by,

$$\psi_{n,\ell}(t, \rho, \Omega) = \frac{1}{N_{\Delta_\phi n\ell}} e^{-iE_n\ell} Y_{\ell,\ell}(\Omega) \left[ \sin^\ell \rho \cos^{\Delta_\phi + \ell} \rho F_1 \left( -n, \Delta_\phi + \ell + n, \ell + \frac{d}{2}, \sin^2 \rho \right) \right], \quad (5.68)$$

where $E_{n,\ell} = \Delta_\phi + 2n + \ell$ and,

$$N_{\Delta_\phi n\ell} = (-1)^n \sqrt{\frac{n!\Gamma(\ell + \frac{d}{2})\Gamma(\Delta_\phi + n - \frac{d-2}{2})}{\Gamma(n + \ell + \frac{d}{2})\Gamma(\Delta_\phi + n + \ell)}}. \quad (5.69)$$

Using the transformation $\tan \rho = r$ we can write the scalar operator as,

$$\psi_{n,\ell_{\text{orb}}}(t, r, \Omega) = \frac{1}{N_{\Delta_\phi n\ell_{\text{orb}}}} e^{-iE_{n,\ell_{\text{orb}}}t} Y_{\ell_{\text{orb}},\ell}(\Omega) \left[ \frac{r^{\ell_{\text{orb}}}}{1+r^2} \sum_{k=0}^{n} (-n)_{k} (\Delta_\phi + \ell_{\text{orb}} + n)_{k} \left( \frac{r^2}{1+r^2} \right)^{k} \right],$$

$$= \sum_{k=0}^{n} \psi_{k}^{\text{orb}}(t, r, \Omega), \quad (5.70)$$

where,

$$\psi_{k}^{\text{orb}}(t, r, \Omega) = \frac{1}{N_{\Delta_\phi n\ell_{\text{orb}}}} e^{-iE_{n,\ell_{\text{orb}}}t} Y_{\ell_{\text{orb}},\ell}(\Omega) \left[ \frac{r^{\ell_{\text{orb}}}}{1+r^2} \sum_{k=0}^{n} (-n)_{k} (\Delta_\phi + \ell_{\text{orb}} + n)_{k} \left( \frac{r^2}{1+r^2} \right)^{k} \right]. \quad (5.71)$$

\textsuperscript{10} The normalization outside should be $1/2$ and not $1/4$ as used in [15]. We thank Jared Kaplan for confirming this.

\textsuperscript{11} There will also be an overall $\alpha'$ dependent factor which can be absorbed into $c_T$ [19].
Putting (5.71) in (5.67) and carrying out the other integrals we are left with just the radial part of the integral,
\[ \delta E_{\text{orb}}^d = -\mu \int r(1 + c_h \alpha^0 r^{-2n})dr \left[ \sum_{k,\alpha=0}^{n} \frac{\mu E_{n,\ell_{\text{orb}}}^2}{(1 + r^2)^2} \psi_k^{\ell_{\text{orb}}}(r) \psi_\alpha^{\ell_{\text{orb}}}(r) + \partial_r \psi_k^{\ell_{\text{orb}}}(r) \partial_r \psi_\alpha^{\ell_{\text{orb}}}(r) \right] = \mathcal{I}_1 + \mathcal{I}_2, \]
(5.72)
where \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) are the contributions from the first and the second parts of the above integral. The leading \( \ell_{\text{orb}} \) dependence comes from the first part of the integral which is also true for \( n \neq 0 \) case and for any general \( d \) dimensions is given by \( (1/\ell_{\text{orb}})^{(d-2)/2} \). Thus we can just concentrate on the first part of the integral for the leading spin dominance of the energy shifts. Thus the integral \( \mathcal{I}_1 \) can be written as,
\[ \mathcal{I}_1 = -\frac{\mu}{N^2 \Delta_{\phi} \ell_{\text{orb}}^d} \sum_{k,\alpha=0}^{n} \frac{(-n)_k(-n)_\alpha(\Delta_\phi + n + \ell_{\text{orb}})k(\Delta_\phi + n + \ell_{\text{orb}})_\alpha}{(\ell_{\text{orb}} + \frac{d}{2})k(\ell_{\text{orb}} + \frac{d}{2})_\alpha k! \alpha!} \times \int_0^\infty \frac{r(1 + c_h \alpha^0 r^{-2n})dr}{(1 + r^2)^{2 + \Delta_\phi + \ell_{\text{orb}} + k + \alpha} + r^{2\ell_{\text{orb}} + 2k + 2\alpha}}. \]
(5.73)
The \( r \) integral gives,
\[ \int_0^\infty \frac{r(1 + c_h \alpha^0 r^{-2n})dr}{(1 + r^2)^{2 + \Delta_\phi + \ell_{\text{orb}} + k + \alpha}} = \frac{\Gamma(1 + \Delta_\phi)\Gamma(1 + \ell_{\text{orb}} + k + \alpha) + c_h \alpha^0 \Gamma(1 + h + \Delta_\phi)\Gamma(1 + \ell_{\text{orb}} + k + \alpha - h)}{2\Gamma(2 + \Delta_\phi + \ell_{\text{orb}} + k + \alpha)}. \]
(5.74)
Hence the integral \( \mathcal{I}_1 \) becomes,
\[ \mathcal{I}_1 = -\frac{\mu}{2N^2 \Delta_{\phi} \ell_{\text{orb}}^d} \sum_{k,\alpha=0}^{n} \frac{(-n)_k(-n)_\alpha(\Delta_\phi + n + \ell_{\text{orb}})k(\Delta_\phi + n + \ell_{\text{orb}})_\alpha}{(\ell_{\text{orb}} + \frac{d}{2})k(\ell_{\text{orb}} + \frac{d}{2})_\alpha k! \alpha!} \times \frac{\Gamma(1 + \ell_{\text{orb}} + k + \alpha)\Gamma(1 + \Delta_\phi) + c_h \alpha^0 \Gamma(1 + h + \Delta_\phi)\Gamma(1 + \ell_{\text{orb}} + k + \alpha - h)}{\Gamma(2 + \Delta_\phi + \ell_{\text{orb}} + k + \alpha)}. \]
(5.75)
Using the reflection formula for the \( \Gamma \)-functions we can write,
\[ (-n)_k(-n)_\alpha = (-1)^{k+\alpha} \frac{\Gamma(n+1)^2}{\Gamma(n+1-k)\Gamma(n+1-\alpha)}. \]
(5.76)
Putting in the normalization and performing the first sum over \( \alpha \) we get,
\[ \mathcal{I}_1 = -\frac{\mu(\ell_{\text{orb}} + 2n)^2\Gamma(\ell_{\text{orb}} + \frac{d}{2} + n)}{2\Gamma(\ell_{\text{orb}} + \frac{d}{2})\Gamma(1 - \frac{d}{2} + n + \Delta_\phi)} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(k + \ell_{\text{orb}} + n + \Delta_\phi)}{\Gamma(1 + \ell_{\text{orb}} + k + h)\Gamma(1 + \Delta_\phi)\Gamma(1 + \Delta_\phi)\Gamma(k + 1)} \times \left[ \Gamma(1 + \ell_{\text{orb}} + k)\Gamma(1 + \Delta_\phi)_{3F_2} \left( -n, k, \ell_{\text{orb}} + 1, \ell_{\text{orb}} + n + \Delta_\phi; \ell_{\text{orb}} + \frac{d}{2}, 2 + k + \ell_{\text{orb}} + \Delta_\phi; 1 \right) \right. \]
\[ + c_h \alpha^0 \Gamma(1 + \ell_{\text{orb}} + k - h)\Gamma(1 + \Delta_\phi + h) \times \left. _3F_2 \left( -n, k, \ell_{\text{orb}} + 1 - h, \ell_{\text{orb}} + n + \Delta_\phi; \ell_{\text{orb}} + \frac{d}{2}, 2 + k + \ell_{\text{orb}} + \Delta_\phi; 1 \right) \right]. \]
(5.77)
To the leading order in $\ell_{orb}$ (after the expansion in large $\ell_{orb}$) this is the expression for $\delta E^d_{orb}$ or equivalently $\gamma_{n,\ell_{orb}}$ from the CFT for general $d$ dimensions. Since the $\ell_{orb}$ dependence does not rely on $n$ dependence we can use the $n = 0$ to show the $\ell_{orb}$ dependence from the $\alpha'$ corrections. Thus in (5.77) we put $n = 0$ to get,

$$I_1 = -\frac{\mu}{2\Gamma(1-\frac{d}{2}+\Delta)}\left(\frac{1}{\ell_{orb}}\right)^{\frac{d-2}{2}}\left[1 + ch\alpha'h\left(\frac{1}{\ell_{orb}}\right)^h\right].$$

(5.78)

In IIB string theory $\alpha'$ corrections to the metric start $h = 3$ due to the well known $R^4$ term. Thus the $\alpha'$ corrections contribute at a much higher order, $O(1/\ell^3_{orb})$, in $1/\ell_{orb}$ and hence does not affect the leading order result. In the rest of the chapter, we will consider only the leading order result in $\ell_{orb}$ (set $c_h = 0$) for general $d$ dimensions.

For even $d$ we can calculate $I_1$ for $n = 0, 1, 2, \cdots$ and infer a general $n$ dependent polynomial. Using this polynomial, the leading $n$ dependence of the energy shifts become,

$$\delta E^d_{orb} = -\mu\left(\frac{1}{\ell_{orb}}\right)^{\frac{d-2}{2}}\left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}n^{d/2} + \cdots\right),$$

(5.79)

where $\cdots$ are terms subleading in $n$. For odd $d$ the polynomial realization is less obvious due to factors of $\Gamma(\Delta \phi + \frac{1}{2})$ etc. nevertheless one can show (see fig.2) at least numerically that the leading order $n$ dependence for odd $d$ should also be the same as (5.79). Hence the above form in (5.79) is true for general $d$ dimensions.

Note that we are retrieving only half of the $n$ dependence from (5.79). The other half will come from the definitions of $\ell_{orb}$, its relation with $\ell$ and $\mu$ which is related to the black hole mass via (5.66). The relation between $\ell_{orb}$ and $\ell$ can be derived following [15]. For the orbit state of an object rotating around a static object, $\ell_{orb}$ is related to the geodesic length $\kappa$ that maximizes the norm of the wave function (5.68). We can approximate the hypergeometric function appearing in (5.68) by taking the large $\ell$ approximation in each term of the sum in (5.70),

$$2F_1\left(-n, \Delta \phi + \ell_{orb} + n, \ell_{orb} + \frac{d}{2}, \sin^2 \rho\right)\approx\sum_{k=0}^{\infty} \frac{(-n)_k(\Delta \phi + \ell_{orb} + n)_k}{(\ell_{orb} + \frac{d}{2})_k k!} (\sin^2 \rho)^k = \frac{\ell_{orb}^{\alpha h} \ell_{orb}^{\beta h} \sin \rho}{\ell_{orb}^{\alpha h} \ell_{orb}^{\beta h} \sin \rho} = \cos^{2n} \rho.$$ 

(5.80)

Using this approximation in (5.68) we find the maxima of $\psi^2_{n,\ell_{orb}J}$, which occurs at,

$$\rho = \tan^{-1} \sqrt{\frac{\ell_{orb}}{2n + \Delta \phi}}.$$ 

(5.81)
Now using the relation, $\sinh \kappa = \tan \rho$, we find for large $\ell_{\text{orb}}$\footnote{Here there is a factor of 2 mismatch with the corresponding result given in \cite{15}. We thank Jared Kaplan for pointing out that this was a typo in \cite{15}.},

$$\kappa = \frac{1}{2} \log \left( \frac{4 \ell_{\text{orb}}}{\Delta \phi + n} \right)$$

But our case is that of a double trace primary operator where none of the two objects is static. However in the semi-classical limit the energy shift of the former case is same as that of the primary if the geodesic distance between the two objects is same in both cases. This gives, $\kappa = \kappa_1 + \kappa_2$, where

$$\kappa_1 (= \kappa_2) = \frac{1}{2} \log \left( \frac{4 \ell_1}{\Delta \phi + n} \right).$$

Since we have a composite operator which is like two particles rotating around each other, the conformal dimension of each descendant state gets the maximum contribution when $\ell_1 \approx \ell_2 \approx \ell/2$ and $n_1 \approx n_2 \approx n/2$ for large $\ell$ and $n$. This is why we have $\ell/(\Delta \phi + n)$ inside log in the above equation. From (5.82) and (5.83) we get, $\ell_{\text{orb}} \approx \ell^2/n$ for large $n$. So, the other factor sitting in front of (5.79) is given by,

$$\mu \ell_{\text{orb}} \left( (d-2)/2 \right) = \frac{2 G_N M \pi (d-2)/2 \Gamma \left( \frac{d}{2} \right)}{d-1} n^{d/2} \frac{\ell^{d-2}}{\ell^{d-2}}$$

Here we have used the equation for black hole mass (5.66). Since $M$ relates to the case of one massive static object at the centre, we will put $M = \Delta \phi + n \approx n$. Using equation (5.61) we get the anomalous dimension to be,

$$\gamma(n, \ell) = -\frac{8(d+1)}{c_T(d-1)^2} \frac{\Gamma(d)^2}{\Gamma \left( \frac{d}{2} \right)^2} n^{d} \frac{\ell^{d-2}}{\ell^{d-2}}.$$

which is the same as that found from CFT.

### 5.6 Discussions

We conclude by listing some open questions and interesting future problems:

- One obvious extension of our analysis is to find the full $n$ dependent expression for arbitrary dimensions. We know that closed form expressions exist for even dimensions (see the appendix for the $d = 6$ result) but it will be interesting to see the analogous expressions for odd dimensions. It should also be possible to repeat our analysis for general twists $\tau_m$. We restricted our attention to the case where this was the stress tensor.

- The $n \gg \ell$ case for arbitrary dimensions which we expect to work out in a similar manner following \cite{3}. In this case we needed $\ell/n$ bigger than some quantity which on the holographic side was identified with a gap—this was similar to the discussion in \cite{13}.
means that for operators with $\ell/n$ smaller than the gap, the result would be sensitive to $\alpha'$ corrections. It could be possible that the anomalous dimensions of these operators would not be negative indicating a problem with causality in the bulk. It will be interesting to extend our holographic calculations to $n \gg \ell \gg 1$.

- One should compare our results with those from the numerical bootstrap methods eg. in [7].

- One could try to see if there exists a simple argument for the leading $n$ dependence following the lines of [20].

- It will be interesting to match the CFT and holographic calculations to all orders in $n$ at least for a class of theories (e.g. $\mathcal{N} = 4$ results in [21]).
Bibliography


Chapter 5. *Universal anomalous dimensions at large spin and large twists*


Chapter 6

Conformal bootstrap for critical exponents

6.1 Introduction

The previous chapters 4 and 5 were aimed at gaining some analytical handle on the large spin sector of the CFT spectrum. A fundamental assumption about the approach in these chapters was existence of a “free” theory and a minimal twist stress tensor decoupled from the rest of the higher spin operators in the theory. Needless to say, these are the characteristics of a strongly coupled theory. In the other regime of weak coupling and for all finite dimensions of the gauge group, this assumption is not true, since the higher spin operators are expected to become dominant in this regime along with the stress tensor. The conventional bootstrap technique mentioned in chapters 4 and 5, then relies on the contributions of all these operators in the spectrum even in the $u \to 0$ and $v \to 1$ limit and becomes cumbersome. On the other hand familiar techniques from the perturbative QFT are already available which can successfully deal with the weakly coupled regime. One such technique is the $\epsilon$-expansion introduced by Wilson and Fisher and studied extensively [1, 2, 3, 4] in order to understand critical phenomena. The essential idea is to consider a weakly interacting theory in $4 - \epsilon$ dimensions that goes to the free theory for $\epsilon \to 0$ and to the interacting theory for a non zero value of $\epsilon$. In $4 - \epsilon$ dimensions, the critical theory corresponds to the so-called Wilson-Fisher fixed point which is strongly interacting theory in three dimensions e.g. 3d Ising Model. However, the technique of the $\epsilon$-expansion relies heavily on the machinery of Feynman diagrams and going beyond the first few orders leads to computing numerous Feynman diagrams which makes this approach cumbersome and clumsy.

As introduced briefly in chapter 2, Polyakov [6] derived a set of consistency equations by demanding compatibility between the operator product expansion in conformal field theory and
Chapter 6. Conformal bootstrap for critical exponents

unitarity\(^1\). Specifically, he considered the \(O(n)^2\) model. By looking at the discontinuity in a
time-ordered four point function (reviewed below), and using dispersion relations, he was able
to compute the position space representation (in \(u \to 0, v \to 1\) limit) of the correlation function.
We will frequently refer to this as the *unitarity approach*. This has to be compatible with what
arises on using the OPE which we will refer to as the *algebraic approach*. He found that there
are certain terms which arise in the unitarity approach which are absent in the algebraic one.
By setting these to zero gave a set of equations constraining the anomalous dimensions and
OPE coefficients. The resulting equations in generality turned out to be too difficult to solve.
However, Polyakov showed that in \(4 - \epsilon\) dimensions, the equations can be solved to leading order
in \(\epsilon\). In order to achieve this he needed to make two important assumptions: a) The anomalous
dimension of \(\phi_i\) began at \(O(\epsilon^2)\) and b) Consistency could be achieved by just retaining two
Lorentz spin zero exchange operators the isospin 0 (\(I = 0\)) scalar and the isospin 2 (\(I = 2\)).
With these assumptions the \(O(\epsilon)\) anomalous dimensions for the isospin scalar and the isospin 2
operators matched with the Feynman diagram approach of Wilson’s.

Recently, Rychkov and Tan \([7]\) have shown how using conformal invariance alone, one can obtain
the leading order anomalous dimensions of a large class of operators. This method just uses
three point functions and appears to be quite powerful. However, new ideas are needed to
extend this to sub leading orders. This was one of the main motivations for us to re-examine
Polyakov’s original calculations to see if it can be used to obtain the sub leading terms in \(\epsilon\).

While the work of Rychkov and Tan eliminates\(^3\) the need for assumption (a) listed above, assumption (b) still needs further investigation. For example can one persist with this assumption
up to \(O(\epsilon^2)\)? In the course of examining this issue, we will show that one can also correctly
get the \(O(\epsilon^2)\) anomalous dimensions of the above operators. This points to the fact that the
unitarity based approach may be more powerful than previously realized and could be another
useful way to complement the powerful numerical techniques of the modern conformal bootstrap
program \([8, 9, 10, 11, 12, 13]\).

The chapter is organized as follows. In section 6.2, we review some of the results existing in
the literature about the anomalous dimensions for the operators \(I = 0\) and the \(I = 2\) Lorentz
scalars in the \(O(n)\) models. The next section 6.3 deals with the algebraic approach. This idea
is based on imposing the operator algebra on the position space Green function derived from
the unitarity in momentum space. Demanding consistency, leads to some algebraic equations
involving the anomalous dimensions which can then be solved order by order in \(\epsilon\). Section
6.4 deals with the construction of a unitary and crossing symmetric four point amplitude in
the momentum space. We have taken only two \((I = 0, 2)\) scalar operators of the \(O(n)\) to
demonstrate consistency as in \([6]\). Considering a right branch cut in the direct channel, one
can build up this amplitude which is both unitary and crossing symmetric. In the next section,

\(^1\)A more recent attempt to study consistency from conformal invariance in the \(O(n)\) model is by Petkou\([5]\).
\(^2\)To keep consistency with the notations in \([6]\) we have denoted the rank of the fundamental group by \(n\) instead
of \(N\) as is usually prevalent.
\(^3\)Replacing instead by the assumption that certain multiplets recombine following the equations of motion as
explained in \([7]\).
6.5, we considered the case of mixed four point functions where one of the operators $\phi_i$ has been replaced with $\phi^2 \phi_i$. Appendix C.5.1 deals with the necessary details for the construction of the unitary and crossing symmetric mixed four point amplitude. As we have pointed out, by demanding consistency one can obtain about the leading $\epsilon$ dependence of the anomalous dimensions of the external operators $\phi_i$ and $\phi^2 \phi_i$. In section 6.6, we comment on the large spin operators. We show that the anomalous dimensions for these operators (for $n = 1$) can be recovered from our bootstrap approach by treating the $\phi^2$ operator as the dominating exchange.

**Notation:** We will use the same notation as in [6]. $d$ will denote the conformal dimension of the exchanged operator (and not the space time dimension!) while $\Delta$ will denote the conformal dimension of the seed scalar operator in the four point function. We will sometimes use $a = 4 - \epsilon$.

6.2 Anomalous dimensions in the $O(n)$ model

Let us review the $O(\epsilon^2)$ results that arise from the diagrammatic approach [1]. In the $O(n)$ theory we denote the scalar by $\phi_i$ with $i = 1, 2 \cdots n$. We work in $4 - \epsilon$ dimensions. The anomalous dimension for $\phi_i$ works out to be

$$\gamma_{\phi_i} = \frac{n + 2}{4(n + 8)^2} \epsilon^2 + O(\epsilon^3). \tag{6.1}$$

The anomalous dimension of the Lorentz scalar, isospin zero ($O_0 = \phi_i \phi_i$) and isospin 2 ($O_2 = \phi_i \phi_j - \frac{2}{n} \phi_k \phi_k \delta_{ij}$) operators are given by

$$\gamma_{O_0} = \frac{n + 2}{n + 8} \epsilon + \frac{n + 2}{2(n + 8)^3} (13n + 44) \epsilon^2 + O(\epsilon^3), \tag{6.2}$$

$$\gamma_{O_2} = \frac{2}{n + 8} \epsilon - \frac{(n + 4)(n - 22)}{2(n + 8)^3} \epsilon^2 + O(\epsilon^3). \tag{6.3}$$

The anomalous dimension of certain gradient operators are also known. The main interest is the Lorentz spin-$\ell$, isospin zero case which includes the stress tensor $O_\ell = \phi_i \partial_{\alpha_1} \cdots \partial_{\alpha_\ell} \phi_i$ – trace

$$\gamma_{O_\ell} = \frac{n + 2}{2(n + 8)^2} (1 - \frac{6}{\ell(\ell + 1)}) \epsilon^2 + O(\epsilon^3). \tag{6.4}$$

Notice that for $\ell = 2$ which corresponds to the stress tensor, the anomalous dimension vanishes to the order shown. We will reproduce this result in the large $\ell$ limit using modern techniques.

Now let us turn to the following question. We know that the diagrammatic approach will need a regularization scheme. As a result, it may be expected that a comparison order by order in $\epsilon$ is not meant to give an agreement unless we know the choice of scheme. However, at criticality, the anomalous dimensions are scheme independent. An explicit check up to $O(\epsilon^2)$ is as follows. It is well known that the beta function is scheme independent up to second order. That is\(^4\), writing $\beta(g) = b_1 g^2 + b_2 g^3 + \cdots$, $b_1$ and $b_2$ are scheme independent. The

\(^4\)In this argument, $b_i$’s are independent of $\epsilon$ in any scheme.
fixed point is obtained by solving $\beta(g) + \epsilon g = 0$ order by order in $\epsilon$. Thus the location of the fixed point is scheme independent up to $O(\epsilon^2)$. The general form of the anomalous dimension is $\gamma(g) = a_1 g + a_2 g^2 + \cdots$. With a different scheme we have $\gamma(\tilde{g}) = \tilde{a}_1 \tilde{g} + \tilde{a}_2 \tilde{g}^2 + \cdots$. At the fixed point we have to $O(\epsilon^2)$, $a_1 g_* + a_2 g_*^2 = \tilde{a}_1 g_* + \tilde{a}_2 g_*^2$. Thus $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$ and the anomalous dimensions are scheme independent to $O(\epsilon^2)$. The critical exponents in general are expected to be scheme independent at every order in $\epsilon$. The argument is that once we substitute $g = g_*$ and get an expression in terms of $\epsilon$, the only way the scheme dependence would come would be through redefinition of $\epsilon$ which is not allowed.

From Polyakov’s work the anomalous dimensions of $O_0$ and $O_2$ follow up to $O(\epsilon)$. We will now review this approach of [6] and also extend it to $O(\epsilon^2)$.

### 6.3 The algebraic approach

We will now use the OPE in a certain limit following [6] to derive the position space form of the four point function. We will focus on only the leading spin-0 exchange. Of course, for the most general analysis one needs to consider the full conformal OPE and impose the algebra at the level of the amplitudes. The OPE for the $O(n)$ model is given by,

$$\phi_i(r)\phi_j(0) = \delta_{ij} \frac{1}{2\Delta_0} + f_{0} \frac{1}{2(2\Delta_0 - 2\Delta_0)} O^0 + f_2 \frac{1}{2(2\Delta_0 - 2\Delta_0)} O^2 + \cdots,$$

(6.5)

where the $\cdots$ terms represent the higher order terms either with higher spin or with higher number of fields or both. We will now consider the four point function of the form,

$$G_{ijkl}(r,r',R) = \langle \phi_i(r)\phi_j(0)\phi_k(R)\phi_l(R + r') \rangle .$$

(6.6)

In general $G_{ijkl}(r,r',R)$ will have all the components of the trace, symmetric-traceless and the antisymmetric-traceless parts based on the above decomposition of the OPE. However, for the spin-0 case of interest we have

$$G_{ijkl}(r,r',R) = \delta_{ij}\delta_{kl}G_T(r,r',R) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{n}\delta_{ij}\delta_{kl})G_{ST}(r,r',R),$$

(6.7)

where these functions are given by,

$$G_T(r,r',R) = (rr')^{-2\Delta_0} + f_0^2 (rr')^{2\Delta_0} R^{-2\Delta_0} + \cdots,$$

$$G_{ST}(r,r',R) = f_2^2 (rr')^{2\Delta_2} R^{-2\Delta_2} + \cdots.$$

(6.8)

The trace and the symmetric-traceless parts in the Green’s function will contain the contributions from both $O^0$ and $O^2$. The main point is to compare the power dependence with what follows from the unitarity approach (next section).

\footnote{We thank Slava Rychkov for emphasising this to us.}
The following point does not appear to be spelt out in [6]. We are interested in the Lorentzian signature since we are considering time-ordered correlation functions. We write $x^2 = -x^+ x^- + x_i x_i$ and set $x_i = 0$ where $x$ is either $r$ or $r'$. Then we consider $x^+ \to 0$ and $x^- \to 0$ as otherwise the OPE would be an expansion in terms of twists and operators with the same twists would be expected to be equally important. For further discussions see [10, 11]. As in [6] we will consider $R^2 \gg r^2, r'^2$.

### 6.4 The unitarity based approach

We will now briefly summarize the unitarity conditions arising in [6]. The logic is motivated by what arises in scattering theory. We consider time-ordered correlators. A discontinuity in an amplitude occurs when intermediate states go on shell—when this happens, the amplitude factorizes. In momentum space, the discontinuity arises from the imaginary part of the Feynman propagator as shown in figure 6.1. Thus the discontinuity (in the complex momentum of the exchanged operator) in a four point function can be written as the product of three point functions times the imaginary part of the propagator. By knowing the discontinuity in the complex momentum space, we can use dispersion relation techniques to know the four point function everywhere. Then we can Fourier transform back to position space and compare with what arises in the algebraic approach.

In a conformal field theory, the three point function is fixed up to some overall constants. We start by considering the four point function of scalars. As in [6], we begin by considering only Lorentz scalar exchange. Thus if we define the three point functions as $T_d(q, p, p')$ where $d$ is the dimension of the exchanged operator and $\text{Im} D(q^2)$ as the imaginary part of the two point function, then, the Fourier transform of the discontinuity in the four point function is given by,

$$\text{F.T. Disc}(\langle T\phi(r)\phi(0)\phi(R)\phi(R + r')\rangle) = T_d(p, q) \text{ Im} D(q^2) T_d(q, p').$$

Due to scale invariance, the imaginary part of the propagator takes the form,

$$\text{Im} D(q^2) = \text{const.} \ (q^2)^{d-a/2}. \tag{6.10}$$
where  
\[ a = 4 - \epsilon. \]

Now we consider the \( O(n) \) model with the field \( \phi_i \). As in [6], we assume that consistency between the unitarity approach and the algebraic approach can be obtained using the operators (bilinear in the fields) with Lorentz spin zero and isospins 0 and 2. The s-channel discontinuity is written as

\[
\text{Disc}_q \frac{1}{p^2(q-p)^2} A_{iklm}(q,p,p') = a_0 \delta_{ik}\delta_{lm} T^{(0)}(q,p)T^{(0)}(q,p')\text{Im}D^{(0)}(q) \\
+ a_2 B_{iklm}T^{(2)}(q,p)T^{(2)}(q,p')\text{Im}D^{(2)}(q),
\]

(6.11)

where \( B_{iklm} = (\delta_{il}\delta_{km} + \delta_{im}\delta_{kl} - \frac{2}{n}\delta_{ik}\delta_{lm}) \) and where the antisymmetric rank-4 \( O(n) \) tensor does not play a role since it involves the exchange of odd Lorentz spin. Here the superscripts \( (0), (2) \) represent the isospin of the exchanged operator. \( a_0 \) and \( a_2 \) will turn out to be related to the OPE coefficients.

**Limiting values of \( T_d(p,q) \):**

Unlike appendix A of [6] where the expressions are worked out for 4 dimensions, we will always be in \( 4 - \epsilon \) dimensions. We denote \( q^2 = s, p^2 = v_1, (p-q)^2 = v_2, (p')^2 = w_1, (p'-q)^2 = w_2 \) and following [6] consider \( v_1 \approx v_2 \equiv v \) and \( w_1 \approx w_2 = w \). The limiting values of the above three point function in the limits when,

\[
s(q^2) \ll v(p^2), \quad \text{and} \quad s(q^2) \gg v(p^2). \quad (6.12)
\]

We are assuming that \((p-q)^2 \approx p^2\). Thus in the two limits the three point function takes the asymptotic form,

\[
T_d(p,q) = \frac{1}{v^2} \begin{cases} 
1 & : s \ll v \\
\frac{f_1 v^{-d/2+\Delta+\epsilon/2}}{f_2 s^{2-d/2-\Delta-\epsilon/2} v^{-2+2\Delta+\epsilon}} & : s \gg v 
\end{cases}
\]

(6.13)

where \( \text{Im}D(s) = s^{d-a/2} \) where \( a = 4 - \epsilon \). Further the coefficients \( f_1 \) and \( f_2 \) take the form,

\[
f_1 = L_{d,2\Delta-d} \frac{\Gamma(d/2)^2}{\Gamma(d)} , \quad f_2 = L_{d,2\Delta-d} \frac{\Gamma(2-\Delta-\epsilon/2)^2\Gamma(-1+d+\epsilon/2)}{\Gamma(1+d/2-\Delta)\Gamma(2+d/2-\Delta-\epsilon/2)} \text{Re}[-1^{\frac{1}{2}}(-d-2\Delta-\epsilon)],
\]

(6.14)

where \( L_{d,2\Delta-d} \) can be read off from (C.33). We will redefine the \( a_I \) to absorb these \( L_{d,2\Delta-d} \) in them and ignore these overall coefficients in the subsequent discussions. Similar expressions are found involving \( w \) instead of \( v \). Knowing the discontinuity, we can now compute the amplitude using the dispersion relation

\[
A(s,v,w) = \int_0^\infty \frac{ds'}{s' - s} \text{Disc}_q A(s',v,w).
\]

Furthermore, we will make the amplitude manifestly crossing symmetric by adding the \( t \) and \( u \) channel contributions.
6.4.1 Calculation of the four point functions

Using (6.10) and the limiting values of $T_d(p, q)$ given in (6.13) we can now calculate the four point function in a regime where $v(p^2) \gg w(p'^2) \gg s(q^2)$. We will discuss the regimes separately in $s$, $t$ and the $u$ channel. Note that whereas in the $s$ channel the exchange particle has momentum $q$, the corresponding momentum for the exchanges in the $t$ and the $u$ channel is $p - p'$ and $p + p' - q$ respectively. The full interacting and crossing symmetric amplitude becomes a sum of the individual contributions of the amplitude from the three channels. We have

$$A(q, p, p') = A^{(s)}(q, p, p') + A^{(t)}(q, p, p') + A^{(u)}(q, p, p'). \quad (6.15)$$

6.4.1.1 $s$-channel

The amplitude in the $s$-channel is given by,

$$A^{(s)}(q, p, p') \approx f_{12} \left( \frac{v w}{s} \right)^{d/2 - \Delta - \epsilon/2} \int_s^w \frac{sd}{s'} \frac{1}{s'^d - a/2} + f_{1} f_{2} \left( \frac{w}{v} \right)^{2+2\Delta+\epsilon} \int_v^w \frac{sd}{s'^d - a/2 - \Delta - \epsilon/2} \frac{1}{s'^d - s'^a/2 - 2\Delta - \epsilon}.$$

We have taken the leading term in the two limits $s \ll v$ and $s \gg v$ for the discontinuous part of the three point functions $T_d(p, q)$. One might ask about the relevance of the sub leading terms we have neglected in the analysis. In the analysis and the regime we are interested in, these sub leading terms do not interfere\footnote{For $0 < s' \ll s$ the denominator cannot be written as $\frac{1}{s'^a} \approx \frac{1}{s'}$, so this regime will not interfere with the terms here.}. Till now we have not put any index on the conformal dimensions $d$ implying that this form is general for the exchanges we have considered. The final expression is,

$$A^{(s)}(q, p, p') = g_1 q^{2d-a}(pp')^{2\Delta+\epsilon-d} + g_2 p^{2\Delta+\epsilon-d} p'^{2\Delta+\epsilon+d-a} + g_3 p^{4-a} p'^{4+4\Delta+2\epsilon}, \quad (6.16)$$

where the functions, $g_i$ take the form,

$$g_1 = -\frac{f_1^2}{d - a/2}, \quad g_2 = \frac{f_1^2}{d - a/2} - \frac{f_1 f_2}{2 + d/2 - a/2 - \Delta - \epsilon/2}, \quad g_3 = \frac{f_1 f_2}{2 + d/2 - a/2 - \Delta - \epsilon/2} - \frac{f_2^2}{4 - a/2 - 2\Delta - \epsilon}, \quad g_4 = \frac{f_2^2}{4 - a/2 - 2\Delta - \epsilon}. \quad (6.17)$$

The $f_i$ and $g_i$ written above will come with an additional label $I$ depending on whether the exchange in $\phi_i \phi_i$ for $I = 0$ or $\phi_i \phi_j + \phi_j \phi_i - \frac{2}{a} \phi_i \phi_i$ for $I = 2$. 
6.4.1.2 \textit{t}-channel

In the \textit{t} channel the exchange momentum is \( s = (p-p')^2 \). The ingoing and the outgoing momenta are \( p, q - p \) and \( p', q - p' \) respectively. Thus the arguments of \( T_d(q,p) \rightarrow T_d(p-p', p-q) \). All the other calculations remain the same. One other important difference is that since here we are in the regime \( p^2 \gg p'^2 \gg q^2 \), hence the exchange momentum is always at least \( O(v) \). Thus,

\[
A(s, v, w) = \int \frac{ds'}{s-s'} T(s', v, w) \approx \int_v^{\infty} \frac{ds'}{s} A(s', v, w),
\]  

(6.18) and the lower limits do not apply here. Again using the form of the propagator in (6.10) and the limiting form of \( T_d(q,p) \) in this channel we get,

\[
A^{(t)}(q,p,p') = -\frac{1}{p^4 p'^4} g_4 p^{4-a} p'^{4+4\Delta+2\epsilon}.
\]

(6.19)

6.4.1.3 \textit{u}-channel

Similar to the \textit{t}-channel we have an exchange momentum \( s = (p+p' - q)^2 \) for which we have an analogous expression. This is again,

\[
A^{(u)}(q,p,p') = -\frac{1}{p^4 p'^4} g_4 p^{4-a} p'^{4+4\Delta+2\epsilon}.
\]

(6.20)

6.4.1.4 The full interacting amplitude

Upto now we have not used the precise form of the exchanged operator. As in [6], we will now assume that the exchanged operators are just \( O^{(0)} = \phi_i \phi_i \) and \( O^{(2)} = \phi_i \phi_j + \phi_j \phi_i - \frac{2}{n} O^{(0)} \delta_{ij} \) scalars. Thus depending on what among these operators is getting exchanged the amplitudes all the channels will come with an index \( I \) and moreover the contributions of the \textit{t} and the \textit{u} channels will come dressed with some mixing between these exchanges. We derive these mixing coefficients in appendix (C.2). With these mixing terms, we can write down the full interacting amplitude (with the index \( I \) denoting the particular exchange) for the \( \langle \phi_i \phi_j \phi_k \phi_l \rangle \) as,

\[
A_{ijkl}(q,p,p') = \delta_{ij} \delta_{kl} A_0^{(0)}(q,p,p') + (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{n} \delta_{ij} \delta_{kl}) A^2(q,p,p'),
\]

(6.21) where,

\[
A^I(q,p,p') = a^I [g_1^{I} q^{-4a} (pp')^{2\Delta+\epsilon-d'} + g_2^{I} p^{2\Delta+\epsilon-d'} + g_3^{I} p^{2\Delta+\epsilon+d'-a}] + (a^I g_3^I - b_I) p^{4-a} p'^{4+4\Delta+2\epsilon},
\]

(6.22) and further,

\[
b_I = \sum_{J=0,2} c_{IJ} a_J g_4^I.
\]

(6.23) \( d^I \) is the conformal dimension of the exchanged scalar for \( I = 0, 2 \) respectively and \( c_{IJ}^I \)'s are given in (C.21).
6.4.2 Coordinate space Green function

The next step is to use the four point Green function and Fourier transform it to the coordinate space so that we can use the operator algebra. We begin by writing,

\[
\langle T\phi_i(r)\phi_i(0)\phi_k(R)\phi_l(R+r') \rangle = G(r, r', R) = \delta_{ij}\delta_{kl}G^0(r, r', R) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{n}\delta_{ij}\delta_{kl})G^2(r, r', R) .
\] (6.24)

On the other hand we can also write,

\[
\langle T\phi_i(r)\phi_i(0)\phi_k(R)\phi_l(R+r') \rangle = \delta_{ij}\delta_{kl}(r r')^{-2+\epsilon} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})R^{-4+2\epsilon} + \int \frac{d^4-p}{p^2}d^{4-\epsilon}p'd^{4-\epsilon}q\epsilon^{ipr+ip'r'+iqR}A_{ijkl}(q, p, p') ,
\] (6.25)

where \( A_{ijkl}(q, p, p') \) also has a similar decomposition in terms of the tensor structure of \( O(0) \) and \( O^{(2)} \) scalar exchanges. Comparing the above two expressions we get,

\[
G^0(r, r', R) = (rr')^{-2+\epsilon} + \frac{2}{n}R^{-4+2\epsilon} + \int \frac{d^4-p}{p^2}d^{4-\epsilon}p'd^{4-\epsilon}q\epsilon^{ipr+ip'r'+iqR}A^0(q, p, p') ,
\]

\[
G^2(r, r', R) = R^{-4+2\epsilon} + \int \frac{d^4-p}{p^2}d^{4-\epsilon}p'd^{4-\epsilon}q\epsilon^{ipr+ip'r'+iqR}A^2(q, p, p') .
\] (6.26)

In the regime of interest \( p^2 \gg p'^2 \gg q^2 \) or equivalently in the coordinate space \( r^2, r'^2 \ll R^2 \), we can to the leading order drop the oscillating exponential and adjust the limits of the integrals\(^7\) over \( p \) and \( p' \) to \( q \ll p' \ll p \ll r^{-1}, r'^{-1} \). Further we can replace the limits of the \( q \) integral from 0 to \( R^{-1} \). Thus the Fourier transform takes the form,

\[
\int \frac{d^4-p}{p^2}d^{4-\epsilon}p'd^{4-\epsilon}q\epsilon^{ipr+ip'r'+iqR}A^i(q, p, p') \approx \int_0^{R^{-1}} d^{4-\epsilon}q \times \int_{R^{-1}}^{r'^{-1}} \frac{dp'}{p'}p'^{-\epsilon} \left( \int_0^{1/p'} dp' + \int_0^{1/r'} dp' \right)^{\epsilon} p^{-\epsilon} A^i(q, p, p') .
\] (6.27)

Using the expressions for the integrals given in the appendix (C.3), we get,

\[
G^0(r, r', R) = (rr')^{-2+\epsilon} + \frac{2}{n}R^{-4+2\epsilon} + a_0[g_1^0S_{2\Delta-a,2\Delta+\epsilon-a}(r, r', R) + g_2^0S_{0,2\Delta+\epsilon-a}(r, r', R)] + (a_0g_3^0 - b_0)T_{0,-4+4\Delta+2\epsilon}(r, r', R) ,
\] (6.28)

\(^7\)We are taking the integral measure as \( d^{4-\epsilon}x \propto x^{3-\epsilon}dx \) for \( x \in \{p, p', q\} \).
We will now assume that the conformal dimensions $\Delta$ and $d$ for the external operators $\phi$ and the exchange can be given in an $\epsilon$-expansion of the form,

$$
\Delta = 1 - \frac{\epsilon}{2} + \gamma_\phi \epsilon^2, \quad d^{0,2} = 2 - \epsilon + \delta^{0,2}(\epsilon).
$$

(6.30)

We will use $\gamma_\phi = (n + 2)/(4(n + 8)^2)$ as follows from conformal symmetries of three point functions in [7]. The coefficients of the log $\left( \frac{\eta^2}{\tau^2} \right)$ terms come only from the $T$ terms in the above expression. Hence setting,

$$
a_I g_I^f - b_I = 0, \quad I = 0, 2, \quad (6.31)
$$

where $b_I$ is defined in (6.23), $g_I^f$ defined in (6.17), gives us two relations for the anomalous dimensions for $\mathcal{O}_0$ and $\mathcal{O}_2$. The remaining relations are obtained by matching the coefficients of $R^{-4+2\epsilon}$ from the remaining of $G^0$ and $G^2$. The coefficient of $R^{-4+2\epsilon}$ takes the form,

$$
a_I \left[ \frac{g_I^f}{(4 + 2\delta_I)(\delta_I + \epsilon - 2a\epsilon^2)^2} + \frac{2g_I^f}{(4 - \epsilon)(\epsilon - 4a\epsilon^2)(\delta_I + \epsilon - 2a\epsilon^2)} \right] = \begin{cases} 
\frac{2}{n} : I = 0 \\
1 : I = 2 
\end{cases}
$$

(6.32)

g_I^f$ etc. can be found in (6.17). Also $\delta_I$ and $a_I$ are assumed to have an expansion of the form,

$$
\delta_I(\epsilon) = \alpha_I \epsilon + \beta_I \epsilon^2 + O(\epsilon^3), \quad a_I(\epsilon) = \rho_I \epsilon^3 + \sigma_I \epsilon^4 + O(\epsilon^5).
$$

(6.33)

The reason behind the particular dependence of $a_I \sim O(\epsilon^2)$ is because of the fact that we expect the OPE coefficients to start at $O(1)$. The solution to the above set of equations is given by,

$$
\alpha_0 = -6 \frac{n + 6}{n + 8}, \quad \alpha_2 = -\frac{n + 6}{n + 8}, \quad \beta_0 = \frac{(n + 2)(13n + 44)}{2(n + 8)^3}, \quad \beta_2 = -\frac{(n + 4)(n - 22)}{2(n + 8)^3}.
$$

(6.34)

Thus the expressions for the dimensions for the scalars $\mathcal{O}_0$ and $\mathcal{O}_2$ become,

$$
d^0 = 2 - \epsilon + \frac{n + 2}{n + 8} \epsilon + \frac{(n + 2)(13n + 44)}{2(n + 8)^3} \epsilon^2 + O(\epsilon^3),
$$

$$
d^2 = 2 - \epsilon + \frac{2}{n + 8} \epsilon - \frac{(n + 4)(n - 22)}{2(n + 8)^3} \epsilon^2 + O(\epsilon^3).
$$

(6.35)

These match exactly with eq(6.2)!

**Why did the calculation work to second order?** Here is a plausible argument. The OPE coefficient appears in the calculation as squared. It is reasonable to assume that there are no fractional powers in $\epsilon$ so that the contributions from other operators that we have not considered will begin at least $O(\epsilon^2)$ higher than in our equations, namely because to $O(\epsilon)$ this approach gave the expected answer which is fixed by other considerations [7]. Thus the contributions from
other operators can be expected to start at $O(\epsilon^3)$. For example, in the free theory $\langle \phi\phi\phi^2 \rangle$ is non zero but the higher three point functions $\langle \phi\phi\phi^n \rangle$ for $n > 2$ are zero. To see that this is consistent, it may be necessary to consider mixed correlators as well which we will perform in the following section.

### 6.5 Mixed correlators

We will consider one more example to illustrate this method. We will consider the example for a mixed correlator: $\langle \phi_i(r)\phi_j(0)\phi_k(R)\phi^2\phi_l(R + r') \rangle$. The last operator is a composite one made out of three fundamental fields $\phi$. We will treat this in the same way as for the four point function of four fundamental fields $\phi_i$. The only difference in this case is going to be that we will need here two different sets of OPEs in this case viz. $\phi_i \times \phi_j$ and $\phi_i \times \phi^2 \phi_j$ respectively where in the previous examples we only needed one type of the OPE $\phi_i \times \phi_j$.

#### 6.5.1 OPE for mixed correlators

We will consider separately these two OPEs now. For the $\phi_i \times \phi_j$ case we can write down with two scalars $O^0$ and $O^2$ as follows,

$$\phi_i(r) \times \phi_j(0) = r^{-2\Delta_0} \delta_{ij} + r^{\Delta_0 - 2\Delta_0} f_{00} O^{(0)} \delta_{ij} + r^{\Delta_2 - 2\Delta_0} f_2 O^{(2)}_{ij} + \ldots, \quad (6.36)$$

whereas for the other OPE we firstly will not have the identity term since the two point function vanishes (they are different operators) and secondly there are additional scalars even in the free field limit that gives a non zero three point function and hence should be considered in the OPE itself. Thus,

$$\phi_i(R) \times \phi^2_j(R + r') = r^{\Delta_0 - \Delta_3} g_{02} O^{(0)} \delta_{ij} + r^{\Delta_2 - \Delta_3} g_{22} O^{(2)}_{ij} + r^{\Delta_{40} - \Delta_3} h_{02} O^{(4,0)}_{ij} + r^{\Delta_{42} - \Delta_3} h_{22} O^{(4,2)}_{ij} \quad (6.37)$$

The scalars $O^{(4,0)}$ and $O^{(4,2)}$ are the fourth order scalars of $I = 0, 2$. Note that in the second OPE we have denoted the conformal dimension for the composite operator by $\Delta_3$. We can as well consider the order four scalars in the OPE of $\phi_i \times \phi_j$ as well. Thus for the four point function we get,

$$\langle \phi_i(r)\phi_j(0)\phi_k(R)\phi^2\phi_l(R + r') \rangle = f_{00} g_{02} r^{\Delta_0 - 2\Delta_3} r^{\Delta_0 - \Delta_3} R^{-2\Delta_0} + f_{22} g_{22} r^{\Delta_2 - 2\Delta_3} r^{\Delta_2 - \Delta_3} R^{-2\Delta_2} + \ldots \quad (6.38)$$

where we have neglected the higher order scalars for the time. Let us remind the readers about the notations again. $\Delta_0 = 1 - v\epsilon/2 + O(\epsilon^2)$ and $\Delta_3 = 3(1 - \epsilon/2) + b\epsilon + O(\epsilon^2)$ are the conformal dimensions for $\phi_i$ and composite operator $\phi^2 \phi_i$ and $\Delta_i = 2 + \delta_i$ are the conformal dimensions for the scalars $O^{(0)}$ and $O^{(2)}$ respectively. Consistency should lead to $b = v = 1$ as in [7].
6.5.2 Leading anomalous dimensions for $\phi_i$ and $\phi^2 \phi_i$

Since the construction of the four point function for the mixed correlator is along the same lines as for the four point function of the fundamental scalars $\phi_i$, we defer the relevant details to appendix (C.5) and quote here the final results. Note that (C.58) serves as the constraint which can be solved order by order in $\epsilon$ to get the anomalous dimensions of the external operators. We will assume a form for the dimensions for $\phi_i$ and $\phi^2 \phi_i$ as,

$$
\Delta_{\phi} = 1 - v \frac{\epsilon}{2}, \quad \Delta_{\phi^3} = 3 \left( 1 - \frac{\epsilon}{2} \right) + b \epsilon.
$$

(6.39)

The leading order expansion in $\epsilon$ for (C.58) is supposed to impose constraints on the coefficients $b$ and $v$. The expected answer is $b = v = 1$. Moreover we assume for the anomalous dimensions of the $\phi^2$ exchange as,

$$
d_I = 2 + \rho I \epsilon + \sigma I \epsilon^2, \quad \text{and} \quad a_I = L_I^1 L_I^2 (\nu_I \epsilon^3 + \mu_I \epsilon^4 + \eta_I \epsilon^5),
$$

(6.40)

where $L_I^1$ and $L_I^2$ are the overall factors associated with the three point functions $\langle \phi \phi \mathcal{O} \rangle$ and $\langle \phi (\phi^2 \phi) \mathcal{O} \rangle$ given in (C.33). Using these to expand (C.58), we find that the leading order term does not show any divergence as $(b, v) \to 1$ but the sub leading terms have double poles as $(b, v) \to 1$ in the form of $((v - 1)(1 + v - 2b))^{-1}$. To remove this divergence we assume that the leading term in $a_I$ goes like,

$$
\nu_I = (v - 1)(1 + v - 2b) \xi_I, \quad \text{and} \quad \mu_I = 0,
$$

(6.41)

where $\xi_I$ is the part of $a_I$ without the zeros. With this substitution, it is easy to see that not only the divergence is removed but the leading term in $a_I$ goes to zero as well as we take either $v \to 1$ or $b \to 1$. We have to justify how to see that the leading behavior of the OPE coefficients are double zeros in $b$ and $v$. Note that the overall factor in $\nu_I$ can be expanded to,

$$
(v - 1)(1 + v - 2b) = (b - v)^2 - (b - 1)^2.
$$

(6.42)

Hence the OPE coefficient for the mixed correlator takes the form,

$$
\text{OPE}_{\phi \phi} \times \text{OPE}_{\phi (\phi^2 \phi)} = \frac{((b - v)^2 - (b - 1)^2) \xi_I c_I L_I^1 L_I^2}{2 d_I (2 \Delta - d_I)(\Delta + \Delta_3 - d_I)} = \frac{1}{\epsilon^2} ((b - v)^2 - (b - 1)^2) \xi_I + \cdots.
$$

(6.43)

So if we set $v = 1$ first, then this factor vanishes for all $b$ and $b = 1$ does not have to be imposed as a required condition. It will imply that the consistency condition does not apriori depend on the leading $\epsilon$ dependence of the dimension for the operator $\phi^2 \phi_i$. Instead if we ask the following question, “Given that the anomalous dimension for the $\phi^2 \phi_i$ operator starts with $O(\epsilon)$ with unit coefficient, what can we say about the anomalous dimension for the operator $\phi_i$?” So the idea is to first set $b = 1$ in the calculation. One can readily see that the overall factor develops a
double zero at \( v = 1 \). Thus the OPE expansion begins like,

\[
\text{OPE}_{\phi \phi} \times \text{OPE}_{\phi(\phi^2 \phi)} = \frac{(1 - v)^2}{\epsilon^2} \xi_I + \cdots,
\]

(6.44)

where \( \cdots \) represent terms with higher orders of \( \epsilon \). Since the exchanges \( I = 0, 2 \) scalars are present even in the free limits, then one would be forced to conclude that the leading order behavior of \( a_I \) should be \( O(1) \) term and higher powers of \( \epsilon \). We can see from the above expansion, that the \( O(1/\epsilon^2) \) term vanishes only for \( v = 1 \) which is another way of putting that the dimension of the fundamental scalars do not receive corrections at order \( O(\epsilon) \). Thus a consistency statement can be formulated as,

“Given that the anomalous dimensions for the \( \phi^2 \phi_i \) operator contributes at \( O(\epsilon) \), then we will require the anomalous dimensions of the external scalars \( \phi_i \) to start at \( O(\epsilon^2) \) for a consistent free limit to exist with \( i = 0, 2 \) scalar exchanges.”

For completeness, we also compute the coefficient \( \eta_i \) in the limit when \( (b, v) \to 1 \). As expected these quantities are free from poles in the above limit and given by,

\[
\eta_i = \frac{1}{2} \rho_I (1 + \rho_I)(1 + 2 \rho_I)^3 \sum_{J \neq I} c_{IJ} \xi_J \frac{(1 + \rho_J)}{(1 + 2 \rho_J)^2},
\]

(6.45)

where \( \rho_I \) are the \( O(\epsilon) \) terms in the anomalous dimensions for the \( I = 0, 2 \) scalar exchanges, \( c_{IJ} \) are the mixing coefficients and \( \xi_I \) are the leading order OPE coefficients.

Alternatively we could ask the question, “what happens if we directly started out assuming that the anomalous dimension of the fundamental operator \( \phi_i \) is \( O(\epsilon^2) \)? There is no apriori reason to assume that the two expansions are related. So to be completely generic we will assume that,

\[
a'_I = L^I_1 L^I_2 (\nu_I' \epsilon^3 + \mu_I' \epsilon^4 + \eta_I' \epsilon^5).
\]

(6.46)

Expanding in \( \epsilon \) we find that as before \( \mu_I' = 0 \) and also that the double zero reflects in the OPE coefficients as a factor of \( (b - 1)^2 \). Then the analog of (6.44) is given by,

\[
\text{OPE}_{\phi \phi} \times \text{OPE}_{\phi(\phi^2 \phi)} = \frac{(1 - b)^2}{\epsilon^2} \xi'_I + \cdots,
\]

(6.47)

where \( \nu_I' = (1 - b)^2 \xi_I' \). We can see that for the above product to begin at \( O(1) \) we have to put \( b = 1 + \gamma_3 \epsilon^2 \) and then the sub leading coefficients \( \eta'_I \) are related to the previous \( \eta_I \) (when we set \( b = 1 \) first) as,

\[
\eta'_0 = -\frac{1}{2} \eta_0, \quad \eta'_2 = -\frac{1}{4} \eta_2.
\]

(6.48)

It is not difficult to relate the coefficients \( \xi_I \) and \( \xi'_I \) by comparing with each other up to some overall constants. But the ratio will depend on the the anomalous dimensions of \( \phi_i \) and \( \phi^2 \phi_i \). We did not consider the sub leading terms in \( \epsilon \) for the anomalous dimensions of \( \phi^2 \phi_i \), since we expect higher order/spin corrections to play a role here. The reason is that in this case, the OPE coefficients do not appear as squared.
6.6 $\epsilon$-expansion for large spin operators from bootstrap

For completeness, we discuss the anomalous dimensions of large spin operators using modern methods. For $(n = 1)$ and

$$J_\ell = \phi \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi,$$  \hspace{1cm} (6.49)

the anomalous dimension is given by,

$$\gamma_{J_\ell} = \left(1 - \frac{6}{\ell(\ell + 1)}\right) \frac{\epsilon^2}{54} + O(\epsilon^3).$$  \hspace{1cm} (6.50)

The spin independent part comes from the anomalous dimensions of the scalar $\phi$. The spin dependent part is given by,

$$\gamma_{J_\ell} = -\frac{\epsilon^2}{9\ell^2}.$$  \hspace{1cm} (6.51)

We will now reproduce this from the results of [9]. Since we know that the approximate conformal blocks do not depend on the dimension $d$ [9, 12], we can use them to solve for the leading anomalous dimensions for large $\ell$ operators with twists $\tau = 2\Delta_\phi$ [9, 12]–we use the normalizations in [9],

$$\gamma(0, \ell) = \frac{\gamma_0}{\ell^m}, \text{ where } \gamma_0 = -\frac{2P_m\Gamma(\Delta_\phi)^2\Gamma(\tau_m + 2\ell_m)}{\Gamma(\Delta_\phi - \frac{2}{n})^2\Gamma(\frac{\tau_m}{2} + \ell_m)^2}.$$  \hspace{1cm} (6.52)

Considering the case where the scalar $\phi^2$ is the dominant exchange, we can set,

$$\tau_m = d - 2 + \epsilon \gamma_{\phi^2}.$$  \hspace{1cm} (6.53)

The scalar $\phi$ also acquires anomalous dimension,

$$\Delta_\phi = \frac{d - 2}{2} + \epsilon^2 \gamma_\phi,$$  \hspace{1cm} (6.54)

although these are at a higher order in $\epsilon$ than those of $\phi^2$ operator. Plugging this in (6.52), setting $\ell_m = 0$ and using the mean field $P_m = 2$ [9], we find that,

$$\gamma_0 = -\frac{\epsilon^2}{9}; \text{ so } \gamma(0, \ell) = -\frac{\epsilon^2}{9\ell^2} + O(\epsilon^3),$$  \hspace{1cm} (6.55)

Instead if we put in the form of the $\gamma_\phi$ for the $O(n)$ models as given in [7], we should get (6.4). For the stress tensor exchange (or any corresponding higher spin exchange with the same twist), $\tau_m = d - 2$. The leading contribution of $\Gamma(\Delta_\phi - \frac{2}{n})$ in the denominator in (6.52), starts with $1/\epsilon^4$ while all the other terms are $O(1)$. Thus the contribution to $\gamma_0$ from these operators begins at a higher order, namely $\sim \epsilon^4$.

Clearly this is more sub leading than the $\phi^2$ exchange. Also since the higher three point functions of the form $\langle \phi \phi \phi^n \rangle$ for $n > 2$ are all zero in the free limit we expect their contributions to be
suppressed further by $O(\epsilon^2)$ compared to the leading $\phi^2$ exchange. With this, one can see that
the higher spin operators of the form in (6.49) are indeed given as in (6.50) to the leading order.
It may be possible to use the results of [12] to get the anomalous dimensions of the large spin
and large twist operators. It will also be interesting to see if we can reproduce these results
using Polyakov’s approach.

6.7 Concluding comments

We have just scratched the tip of the iceberg. It will be important and interesting to understand
the emergence of the consistency conditions that Polyakov has derived in the second half of his
paper and see if they can be solved or understood even if some approximate sense—for example,
can we understand the systematic of higher order or higher spin operators? This approach
seems to have its own merit as we have demonstrated through the $\epsilon$-expansion. Apart from
this, it will be interesting to reinterpret Polyakov’s work [6] in terms of the modern bootstrap
language. It will be also interesting to extend the formalism to other dimensions [14].
Bibliography


Chapter 6. *Conformal bootstrap for critical exponents*


Chapter 7

Holographic stress tensor at finite coupling

7.1 Introduction

The previous three chapters 4, 5 and 6 dealt with analytical aspects of the conformal bootstrap program and how one can use the knowledge to their advantage to put constraints on certain regions of the CFT spectrum. While chapters 4 and 5 primarily were concerned with the large spin sector of the spectrum for a strongly coupled large $N$ CFT, chapter 6 focused on the perturbative regime, shedding some light on the anomalous dimensions of low lying operators of the $O(n)$ models with finite $n$. However in all these examples we considered the simplest case of scalars or the fields transforming in the fundamental representation of the internal symmetry group (e.g. in chapter 6 we considered fundamental fields $\phi_i$).

However, the complete knowledge of the CFT spectrum is incomplete from the above analysis since neither the OPE of the singlet scalars nor the OPE of the fundamentals of $O(n)$ group capture all the operators that appear in the spectrum. For this we will need other set of correlators (for e.g the higher spin correlators and the mixed correlators therein) where these additional operators show up. As a first step, one can compute the correlators of the spin−1 currents or stress tensors. But even from the CFT side, the calculation of these correlators is much more difficult than the scalar case mainly because of increasing complexities of the bootstrap equations and multiple equations depending on the tensor structures of the external spinning operators. On the other hand, from the bulk point of view, these correlators are equivalent to the scattering amplitudes of either gauge fields or gravitons and computing these (using Witten diagrams) are tedious even at the tree level. Of these, the graviton scattering is particularly of interest. This computation is tedious and as a first step towards this, we might calculate the three point graviton vertex in the bulk. By this we are actually calculating the three point functions of gravitons in the bulk.
In this chapter we will compute explicitly the three point functions of the gravitons (which corresponds to the three point functions of the stress tensors in CFT) in presence of generic higher derivative (curvature) terms in the bulk gravity action. Note that while the generic two derivative gravity in most holographic approaches [1] corresponds to infinite ’t Hooft coupling and an infinite number of colors, we expect from the context of the canonical example of $\mathcal{N} = 4$ SU(N) SYM with the gravitational dual being type IIB superstring theory on AdS$_5 \times$ S$^5$, that finite coupling corrections should correspond to specific higher derivative ($\alpha'$) corrections in the low energy effective action of the theory. In addition to these corrections, there are also non-local contributions, for example from graviton loops. One essential step to take into account the contributions at finite coupling, is to be able to compute the holographic stress tensor and its correlation functions for an arbitrary higher derivative theory of gravity.

To progress, we start by calculating the holographic stress tensor itself from first principles [2] which appears to be prohibitively difficult, since the generalized Gibbons-Hawking term and often counter terms are not known for an arbitrary higher curvature theory, which stymies any progress using conventional approaches and only some sporadic results for specific examples are known in the literature [3, 4]. A way around this problem is using the first law of entanglement pertaining to spherical entangling surfaces [5]. This works as follows: The first law of entanglement states that

$$\Delta S = \Delta H,$$

(7.1)

where for two density matrices $\rho, \sigma$ with $\sigma \equiv e^{-H}/\text{tr } e^{-H}$ being the reduced density matrix for a spherical entangling surface in a CFT with $H$ being the modular hamiltonian, $\Delta H = \langle H \rangle_\rho - \langle H \rangle_\sigma$ and $\Delta S = S(\rho) - S(\sigma)$ with $S(\rho) = -\text{tr } \rho \log \rho$ being the von Neumann entropy for $\rho$ and is the entanglement entropy for a reduced density matrix $\rho$. The equality arises at linear order in perturbation, meaning that $\rho, \sigma$ belong to some family of density matrices $\hat{\rho}$ parametrized by some perturbation parameter $\lambda$ such that $\sigma = \hat{\rho}(\lambda = 0), \rho = \hat{\rho}(\lambda)$ and we are interested only in linear order in $\lambda$. At nonlinear order in $\lambda$ we get an inequality which corresponds to the positivity of relative entropy, leading to $\Delta H > \Delta S$. The expression for $H$ (which will be given later) involves the time-time component of the field theory stress tensor. In holography, for spherical entangling surface, the entanglement entropy for the vacuum state across the sphere $S^{d-2}$ gets mapped to the thermal entropy on $R \times H^{d-1}$. Using the gravitational dual, the thermal entropy is computed using the Wald entropy which is known for an arbitrary higher derivative theory of gravity. For bifurcate Killing horizons, there is a theorem due to Iyer and Wald [7] which states that linearized perturbations satisfy the first law of thermodynamics which translated to our case means that eq.(7.1) would be applicable with linearized perturbations in the Wald formula. Thus the LHS of eq.(7.1) can be computed using the linearization of the Wald formula. The RHS of eq.(7.1) has the perturbation of the time-time component of the field theory stress tensor which now can be determined. In order to be able to do the integral, one approximates the entangling surface radius $R$ to be small. Since the only dimensionless parameter is $R^d \langle T_{\mu\nu} \rangle$, the perturbative expansion can be done either by treating $R$ to be small or by treating $\langle T_{\mu\nu} \rangle$ to be small. Thus although the expression
for the stress tensor obtained using the above logic pertains to an excited state that is a small
perturbation from the vacuum state, the expression should hold for any \langle T_{\mu\nu} \rangle. Since this can be
done for an arbitrary higher derivative theory, we thus know how to compute the holographic
stress tensor for such a bulk dual.

The result of this calculation is a very compact expression for the stress tensor in terms of
certain parameters appearing in the linearized expression of the Wald formula. In particular,
if one ignores covariant derivatives of the curvature tensor, the result can be worked out quite
simply. Writing the linearized Wald functional as

\[ \delta E_{abcd} = -c_2 g^{(ab}g^{cd)} h - c_3 h^{(ab}g^{cd)} + c_4 g^{(ab}g^{cd)} R + c_5 R^{(ab}g^{cd)} + c_6 R_{abcd}, \]  

(7.2)

where \( c_2 \) and \( c_3 \) are not independent coefficients but related to the other coefficients as,

\[ c_2 = -2d c_4 - c_5, \quad c_3 = 2c_1 - (d - 1)c_5 - 4c_6, \]  

(7.3)

which we will demonstrate later, \( c_1 \) is defined through the Wald function \( E_{abcd} = c_1 g^{(ab}g^{cd)} \) and

where,

\[ \delta g^{ab} = -h^{ab}, \quad \delta R_{abcd} = R_{abcd}, \quad h = g_{cd}h^{cd}, \]  

(7.4)

one finds that

\[ \langle T_{\mu\nu} \rangle = d\tilde{L}^{d-3} [c_1 + 2(d - 2)c_6] h_{\mu\nu}^{(d)}, \]  

(7.5)

where \( h_{\mu\nu}^{(d)} \) appears in the Fefferman-Graham expansion of the asymptotic AdS metric as

\[ ds^2 = \tilde{L}^2 \frac{dz^2}{z^2} + \frac{1}{z^2} (g_{\mu\nu}^{(0)} + z^2 g_{\mu\nu}^{(2)} + \cdots z^d h_{\mu\nu}^{(d)} + \cdots) dx^\mu dx^\nu. \]  

(7.6)

\( \tilde{L} \) is the AdS radius. This begs the question: What is this simple proportionality constant
depending on \( c_i \)'s in (7.5)? Since the linearized Wald functional was involved in the derivation
of this simple form, with hindsight we can anticipate that there are simplifications waiting
to happen if we consider rewriting the Lagrangian as a background field expansion around
a suitable background. Recently this background field method has been used to find simple
expressions for trace anomalies in even dimensions [8]. We will make a simple modification to
this method so that the anomaly calculation can be carried out easily using Mathematica. Let us
now explain why this method is useful in correlating with the results above as well as calculating
higher point correlation functions. Given a Lagrangian \( \mathcal{L}(g_{ab}, R_{bcde}, \nabla_f R_{bcde}, \cdots) \), we are going
to treat \( g_{ab} \) and \( R_{abcd} \) as independent variables. We are going to expand \( R_{abcd} \) around \( \bar{R}_{abcd} = -\frac{1}{\tilde{L}^2} (g_{abcd} - g_{ad}g_{bc}) \) where \( g_{ab} \) in this expression is the full metric. Raising and lowering indices
and the covariant derivative is done using the full metric. Define \( \Delta R_{abcd} = R_{abcd} - \bar{R}_{abcd} \). Then
on the AdS background \( (g_{\mu\nu}^{(0)} = \eta_{\mu\nu}) \) this quantity is zero. Further if we linearize this then in

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1 Note that we are considering field theory in flat space.
2 The normalization of \( \Delta H \) is fixed by the definition of modular Hamiltonian in (7.1). On the RHS the
   normalization of \( \Delta S \) is fixed by holography where we demand that the definition of \( S \) gives the correct universal
terms. This resolves any ambiguity in the definition of \( \langle T_{\mu\nu} \rangle \) in (7.5) by using \( h_{\mu\nu}^{(d)} \) in (7.6).
the transverse traceless gauge, it can be easily checked that \((\Delta R_{ab})^L = (\Delta R)^L = 0\). This is the reason why the expressions we will compute for the stress tensor correlation functions will take on simple forms. Let us start with \(L(g^{ab}, R_{bcde})\), i.e., no covariant derivatives (we will set the AdS radius \(\tilde{L} = 1\) from hereon and will reinstate it when needed). The Lagrangian after doing the background field expansion takes the form

\[
L = (c_0 + c_1 \Delta R + \frac{c_4}{2} \Delta R^2 + \frac{c_5}{2} \Delta R^{ab} \Delta R_{ab} + \frac{c_6}{2} \Delta R^{abcd} \Delta R_{abcd} + \sum_{i=1}^{8} \tilde{c}_i \Delta K_i + \cdots),
\]

where \(c_0 = -2dc_1\) from lowest order equations of motion and \(\Delta K_i = K_i |_{R \rightarrow (R - \bar{R})}\). Note that we are not treating \(c_i\)’s perturbatively. This Lagrangian can be shown to lead to \((7.2)\). The basis for the third order terms is given by

\[
K_i = (R^3, R^a_i R^b_i R^c_i, R R^{ab} R_{ab}, R R^{abcd} R_{abcd}, R_{ab} R^{abcd} R_{cde},
R_{abcd} R^{ab} R^{cd} R_{ef}, R_{abcd} R^{acde} R^{b} R_{ef}),
\]

(7.8)

We are not using an explicit overall factor of \(1/2\ell_p^{d-1}\) with the action since all the coefficients in the action are assumed to implicitly have the factor. In order to compute \(n\)-point functions we expand the bulk action up to \(n\)th order in the perturbation. However, since \(\Delta R_{abcd}^{AdS} = 0\), this means that we only need to retain up to \(O((\Delta R)^n)\) terms in the Lagrangian. Thus, the background field expanded Lagrangian is an expansion in terms of the correlation functions of the stress tensor. Further simplifications happen. Consider \((\Delta R)^2\) or \((\Delta R_{ab})^2\). Since the linearized \(\Delta R\) and \(\Delta R_{ab}\) both vanish, these terms can only contribute to four-point functions onwards. Thus we do not expect \(c_4\) or \(c_5\) in eq.(7.7) above to enter the one, two or three point functions. This is consistent with the absence of these coefficients in eq.(7.5). Moreover, this conclusion does not change on including covariant derivatives.

Let us now summarize our findings for the correlation functions that follow from the Lagrangian in eq.(7.7). The stress tensor two point function takes the form

\[
\langle T_{ab}(x)T_{cd}(x') \rangle = \frac{C_T}{|x - x'|^{2d}} I_{ab,cd}(x - x'),
\]

(7.9)

where

\[
I_{ab,cd}(x) = \frac{1}{2} [I_{ab}(x) I_{cd}(x) + I_{ad}(x) I_{bc}(x)] - \frac{1}{d} \eta_{ab} \eta_{cd},
\]

(7.10)

and

\[
I_{ab}(x) = \eta_{ab} - 2 \frac{x_a x_b}{x^2}.
\]

(7.11)

The coefficient \(C_T\) from the \(d + 1\) dimensional bulk Lagrangian works out to be

\[
C_T = f_d \tilde{L}^{d-1} [c_1 + 2(d - 2) c_6],
\]

(7.12)
where $\tilde{L}$ is the AdS radius and $f_d$ is constant $d$ dependent factor given by \[ f_d = 2^{d+1} \frac{\Gamma(d+1)}{d-1 \pi^{d/2} \Gamma(d/2)}. \] (7.13)

Thus the holographic stress tensor in eq. (7.5) can be written as

\[ \langle T_{\mu\nu} \rangle = \frac{d}{L^2 f_d} C_T h^{(d)}_{\mu\nu}. \] (7.14)

Further for even dimensional CFTs the coefficient $C_T$ is related to a B-type anomaly coefficient as we will show. In particular, the A-type Euler anomaly coefficient is simply proportional to $c_1$ while the B-type anomaly coefficient (conventionally called $c$ in $4d$ and $B_3$ in $6d$) is proportional to $C_T$.

We use the method of background field expansion to calculate the three point functions of stress tensor. Following the simple method devised in [10, 11] and used in [12] we perform the calculation of the three point function in a shock wave background and obtain information about the three point function from the energy flux given by (these results are for $d \geq 4$, for $d = 3$, the term proportional to $t_2$ is absent),

\[ \langle t(n) \rangle = \frac{E}{4\pi \Omega_{d-2}} \left[ 1 + t_2 \left( \frac{\epsilon_{ij}\epsilon_{ik}\eta^j\eta^k}{\epsilon_{ij}\epsilon_{ij}} - \frac{1}{d-1} \right) + t_4 \left( \frac{\epsilon_{ij}\eta^j\eta^k}{\epsilon_{ij}\epsilon_{ij}} \right)^2 - \frac{2}{d^2 - 1} \right], \] (7.15)

where $\Omega_{d-2}$ is the volume of a unit $(d - 1)$ sphere and,

\[ t_2 = \frac{d(d-1)}{c_1 + 2(d-2)c_6}[2c_6 - 12(3d + 4)\tilde{c}_7 + 3(7d + 4)\tilde{c}_8], \]
\[ t_4 = \frac{6d(d^2 - 1)(d+2)}{c_1 + 2(d-2)c_6}(2\tilde{c}_7 - \tilde{c}_8). \] (7.16)

$n$ is the unit normal in the direction in which energy flux is measured and $t_2$ and $t_4$ are determined holographically. The coefficients $t_2$, $t_4$ and $C_T$ are related to the three independent coefficients appearing in the three point functions [13, 14] \(^3\). Notice that for $d = 4$, the $\tilde{c}_7, \tilde{c}_8$ dependence in $t_2$ and $t_4$ are packaged in the same way, namely as $2\tilde{c}_7 - \tilde{c}_8$. This is indicative of the fact that $c, a, t_2, t_4$ satisfy the relation $(c - a)/c = t_2/6 + 4t_4/45$. This relation enables one to extract the 4d Euler anomaly $a$ from the knowledge of two and three point functions. In six (and higher) dimensions, there is no such relation (in fact not even for a linear combination of the A-anomaly and the B-anomaly coefficients) indicating the fact that a similar relation involving the Euler anomaly coefficient will also involve higher point correlation functions.

\(^3\)The relation between $t_2$, $t_4$ and $C_T$ and the CFT coefficients $A$, $B$ and $C$ are

\[ C_T = \frac{\Omega}{2d(d+2)}[(d-1)(d+2)A - 2B - 4(d+1)C], \]
\[ t_2 = \frac{2(d+1)(d-2)(d+1)A + 3d^2B - 4d(d+1)C}{(d-1)(d+2)A - 2B - 4(d+1)C}, \]
\[ t_4 = \frac{d+1(d+2)(2d^2 - 3d - 3)A + 2d^2(d+2)B - 4d(d+1)(d+2)C}{(d-1)(d+2)A - 2B - 4(d+1)C}. \] (7.17)
We can easily extend the above results to the $L(g^{ab}, R_{bcde}, \nabla f R_{bcde})$ case, i.e., to the situation where there are at most two covariant derivatives of the curvature tensor in the action. First notice that since the linearized $\Delta R_{ab}$ and $\Delta R$ both vanish, only terms like $\nabla e \Delta R_{abcd} \nabla e \Delta R_{abcd}$ will contribute to the two and three point functions while only terms like $\Delta R_{....} \nabla \Delta R_{....} \nabla \Delta R_{....}$ will contribute to the three point functions. Further, we will show that using the Bianchi identities and integration by parts [15], the $\nabla e \Delta R_{abcd} \nabla e \Delta R_{abcd}$ terms can be rewritten in terms of $(\Delta R_{....})^3$ and $\nabla a \Delta R_{bc} \nabla a \Delta R_{bc}$. Since the last two terms do not contribute to two or three point functions, the result for the two point functions will involve a redefined $c_6$. We will explicitly show that the result for the three point functions also follows a similar simple trend.

As an application for our methods we will compute the ratio of shear viscosity ($\eta$) to entropy density ($s$) for a general four derivative bulk dual, without assuming the coupling constants to be small (for earlier related work see [16]). Then following [12], we will demand that $-3 \leq t_2 \leq 3$ as well as $C_T > 0$, $s > 0$. These constraints were sufficient in the Gauss-Bonnet case [10, 17] to lead to $\eta/s \geq 16 \frac{1}{25} \approx 0.641 \frac{1}{4\pi}$. We will find that in the general four derivative case, we can tune the couplings so that these conditions are satisfied but $\eta/s$ is arbitrarily small. This is of course due to the fact that the underlying theory has non-unitary modes. We will also show that for the Weyl-squared theory, the above constraints lead to $\eta/s \geq 12 \frac{1}{17} \approx 0.706 \frac{1}{4\pi}$ while including Weyl-cubed terms, the same constraints lead to $\eta/s \geq 0.17 \frac{1}{4\pi}$. Both these theories will have non-unitary modes supported near the horizon. It is interesting to note that there is still a bound on the ratio in such theories.

The rest of the chapter is organized as follows. We start with an introduction to the first law of entanglement in section (7.2). In section(7.3.1) we give a brief review of the calculations of [5]. In section(7.4.1) we calculate the holographic trace anomalies by considering the background field expanded Lagrangian and also how various coefficients of the Lagrangian in [5] are related to the Lagrangian we are considering. We then compute the trace anomalies in $d = 2, 4, 6$ and show that the B-type anomalies are the coefficients in the expression for the holographic stress tensor. In section(7.4.2) we extend the analysis to Lagrangians containing covariant derivatives on the Riemann tensors and show how the anomaly coefficients get modified. More specifically we show that $c_6$ in the B-type anomaly coefficients can be replaced by an effective $c_6'$ in presence of the $\nabla R$ terms in the Lagrangian. In section (7.5) and section(7.6) we extend the analysis to calculating the holographic two and three point functions of the stress tensor. We show that the coefficient in the holographic one point function of the stress tensor is related to the coefficient of the holographic two point functions of the stress tensor for arbitrary dimensions. Section (7.7) presents one application of the method of background field expansion in the calculation $\eta/s$. We present the calculations for Weyl-squared, Weyl cubed and general $R^2$ gravity (appendix (D.5)). We also show that the bounds for $\eta/s$ for these theories pertaining to the physical constraints satisfied by the three point functions are much smaller that the KSS bound [18]. We end with a discussion about open problems in section(7.8).
7.2 First law of entanglement

This section will shed some light on the basics of the first law of entanglement which is the basis of the rest of the discussions in this chapter. To start with, for a generic QFT separated into two subsystems $A$ and $\bar{A}$, the state of the subsystem $A$ is defined through the partial trace of the reduced density matrix over the degrees of freedom in $\bar{A}$. Thus,

$$\rho_A = \text{tr}_{\bar{A}} \rho_{\text{tot}}, \quad \rho_{\text{tot}} = A \oplus \bar{A}. \tag{7.18}$$

The corresponding entanglement entropy for the subsystem $A$ is given by the usual Von-Neumann entropy,

$$S_{EE}(A) = -\text{tr} \rho_A \log \rho_A. \tag{7.19}$$

Using further the hermiticity and the positive (semi) definiteness properties of the density matrix, we can write this $\rho_A$ in the form,

$$\rho_A = \frac{e^{-H_A}}{\text{tr} \ (e^{-H_A})}, \quad \text{tr} \ (\rho_A) = 1, \tag{7.20}$$

where $H_A$ is the generating function for the reduced density matrix and is also called the Modular Hamiltonian. The first order variation of the entanglement entropy (due to excitation of the vacuum state in $A$) is given by,

$$\delta S_{EE}^A = \text{tr} \ (\delta \rho_A H_A) = \delta \langle H_A \rangle, \tag{7.21}$$

where we have used $\text{tr} \ (\delta \rho_A) = 0$. In general the modular hamiltonian is difficult to ascertain for any general given subsystem. However, in simple enough cases (such as a spherical entangling region) this modular hamiltonian takes a simple form. Again, if somehow the given state can be mapped on to a thermal state for which $\rho = e^{-\beta H_\tau}/(\text{tr} \ (e^{-\beta H}))$ then, the above equation for the first order variation of the entropy becomes,

$$\delta \langle H \rangle = T \delta S_{EE}^A, \tag{7.22}$$

where $T$ is the temperature of the thermal state. This is called the first law of entanglement which can be interpreted as the quantum version of the first law of thermodynamics.

As a simple example of finding the modular hamiltonian for a ball shaped region on the boundary, $B(x_0, R)$ centred at $x_0$ with radius $R$, first note that the ball can be mapped on to a thermal state residing on $\mathbb{R} \times H^{d-1}$ through a conformal transformation. The density matrix for the thermal state is,

$$\rho_\tau = \exp(-2\pi R H_\tau), \tag{7.23}$$

where $H_\tau$ is the hamiltonian (conserved charge) corresponding to the dilatation operator $\partial_\tau$. To convert the thermal state into the $T = 0$ state on the ball, we apply an inverse conformal
transformation on $\mathbb{R} \times H^{d-1}$ so that,

$$2\pi R \partial_r \to \zeta^\mu_B \partial_\mu.$$ \hfill (7.24)

Note that the conformal transformations relating $B(x_0, R)$ with $\mathbb{R} \times H^{d-1}$ is given by,

$$t = R \frac{\cosh \frac{\tau}{R}}{\cosh \mu + \cosh \frac{\tau}{R}}, \quad r = R \frac{\cosh \mu}{\cosh \mu + \cosh \frac{\tau}{R}},$$ \hfill (7.25)

where $R$ is the radius of $B(x_0, R)$ and $r^2 = |x_i - x_0^i|^2$. From this, one can see that,

$$\zeta^\mu_B = \frac{i\pi}{R} [R^2 P_t + K_t],$$ \hfill (7.26)

where,

$$iP_t = \partial_t, \quad \text{and} \quad iK_t = -(t - t_0)^2 + |x_i - x_0^i|^2 \partial_t - 2(t - t_0)(x_i - x_0^i) \partial_i.$$ \hfill (7.27)

And finally mapping of the Hamiltonian from $\mathbb{R} \times H^{d-1}$ to the ball $B(x_0, R)$ is,

$$H_B = \int_S d\Sigma^\mu T_{\mu\nu} \zeta_B^\nu,$$ \hfill (7.28)

which gives for the ball on a constant $t = t_0$ slice on $S$,

$$H_B = 2\pi \int_{B(x_0, R)} d^{d-1}x \frac{R^2 - |\vec{x} - \vec{x}_0|^2}{2R} T_{tt}(t_0, \vec{x}),$$ \hfill (7.29)

and lastly,

$$E_B = \langle H_B \rangle = 2\pi \int_{B(x_0, R)} d^{d-1}x \frac{R^2 - |\vec{x} - \vec{x}_0|^2}{2R} \langle T_{tt}(t_0, \vec{x}) \rangle.$$ \hfill (7.30)

The first law thus implies $\delta S^{B}_{EE} = \delta E_B$. For CFTs with holographic duals, the above mapping from the ball to the thermal states corresponds in the bulk to the mapping of the empty AdS to a hyperbolic BH (i.e the AdS-Rindler patch) for which the calculation of the entanglement entropy boils down to the calculation of the BH entropy using the Wald entropy functional for simple enough entangling surfaces on the boundary (such as a sphere) since the difference of the entanglement functional and the Wald functional (which is proportional to the squares of the extrinsic curvatures) vanish. Thus,

$$\delta S^{B}_{EE} = \delta S^{Wald},$$ \hfill (7.31)

from the holographic side. On the other hand, the analog of the energy variation, in the bulk is,

$$E^{grav}_B = \langle H_B \rangle = 2\pi \int_{B(x_0, R)} d^{d-1}x \frac{R^2 - |\vec{x} - \vec{x}_0|^2}{2R} \langle T^{grav}_{tt}(t_0, \vec{x}) \rangle,$$ \hfill (7.32)

which is proportional to the one point function of the stress tensor in the bulk. Thus to sum up the calculation of the first law of entanglement in the bulk,

$$\delta S^{Wald} = \delta E^{grav}_B.$$ \hfill (7.33)
In the successive sections of this chapter, we will analyze this relation in more detail and observe the far reaching consequences of this equality.

### 7.3 Stress tensor from first law of entanglement

In this section we review the derivation of the holographic stress tensor from the first law of entanglement [6] for (7.7) following [5]. The modular Hamiltonian for a spherical entangling region of radius \( R \) and centered around \( x = x_0 \), is given by

\[
H_A = 2\pi \int_{A(R,x_0)} d^{d-1}x \frac{R^2 - |x - x_0|^2}{2R} \langle T_{tt} \rangle, \tag{7.34}
\]

and for any perturbation around the CFT vacuum we have

\[
\Delta H_A = 2\pi \int_{A(R,x_0)} d^{d-1}x \frac{R^2 - |x - x_0|^2}{2R} \delta T_{tt}. \tag{7.35}
\]

As mentioned in [5], the entanglement entropy of the spherical entangling region in the vacuum CFT is equal to the entropy of a thermal CFT on a hyperbolic cylinder with the temperature set by the length scale of the hyperbolic space time. From the holographic side the thermal entropy is given by the entropy of the hyperbolic black hole, which for any classical higher derivative theory of gravity is evaluated using the Wald formula [7]

\[
S_{Wald} = -2\pi \int_{\mathcal{H}} d^n\sigma \sqrt{h} \frac{\delta \mathcal{L}}{\delta R_{ab}^{cd}} n^{ab} n_{cd}, \tag{7.36}
\]

where \( \mathcal{L} \) is given in (7.7) and \( n^{ab} \) is the unit bi normal to the bifurcate Killing horizon \( \mathcal{H} \). In general the Wald entropy functional differs from the entanglement entropy functional by squares of the extrinsic curvature [19] but for the spherical entangling region these terms vanish and \( S_{Wald} = S_{EE} \) at the linear order in perturbations [5]. Further, the perturbations of the vacuum CFT imply perturbations of the thermal CFT since the perturbations of the vacuum AdS imply perturbations of each of the AdS-Rindler wedges for the thermal state\(^4\).

Before proceeding we will specify the notations and conventions. Throughout we set the AdS radius\(^5\) \( \tilde{L} = 1 \) except where we explicitly restore it on dimensional grounds. \( R \) is the radius of the entangling ball. In terms of the Poincaré coordinates, AdS space time is given by,

\[
ds^2 = \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2}. \tag{7.37}
\]

The spherical entangling region \( A \) in the vacuum CFT is associated with the hemispherical region \( \tilde{A} \) in the black hole background given by \( \tilde{A} = \{ t = t_0, (x^i - x_0^i)^2 + z^2 = R^2 \} \) in Poincaré

\(^4\) This assumption is only valid at the leading order in perturbations. In the next order the hyperbolic horizon changes due to the perturbations and we do not have the AdS-Rindler patch.

\(^5\) Note that \( L \) is the length associated with the cosmological constant.
coordinates. These two different regions have the same boundary $\partial A$ in the boundary CFT. Thus $S_{EE}$ is equal to $S^{Wald}$ evaluated on $\tilde{A}$. Similarly the perturbation $\Delta S_{EE}$ of the vacuum CFT is equal to $\delta S^{Wald}$. For holographic CFTs the gravitational version of $\delta S_A = \delta E_A$ is given by $\delta S^{grav} = \delta E^{grav} = \delta S^{Wald}$ and can be used to relate the $\langle T_{\mu\nu}\rangle$ to the asymptotic form of the metric in the holographic side. In the limit of $R \to 0$, $\delta \langle T_{tt}(t_0,x)\rangle$ can be replaced by its central value $\delta \langle T_{tt}(x_0)\rangle$ and we have using $\delta E_A = \delta S_A$,

$$\delta \langle T_{tt}(x_0)\rangle = \frac{d^2 - 1}{2\pi d_{d-2}} \lim_{R \to 0} \left( \frac{1}{R^d} \delta S^{Wald}_A \right),$$

(7.38)

and repeating for arbitrary Lorentz frames we have

$$u^\mu u^\nu \delta \langle T_{\mu\nu}(x_0)\rangle = \frac{d^2 - 1}{2\pi d_{d-2}} \lim_{R \to 0} \left( \frac{1}{R^d} \delta S^{Wald}_A \right).$$

(7.39)

The variation of the Wald entropy around the hyperbolic black hole background for an arbitrary higher derivative theory of gravity is given by

$$\delta S^{Wald} = \delta \left( -2\pi \int_{\tilde{A}} E^{abcd} R^{\epsilon\ab\cd} \right),$$

(7.40)

where $E^{abcd}$ is the Wald functional of the curvatures and their covariant derivatives, $\epsilon^{ab}$ is the volume element and $n_{cd} = n_1^c n_2^d - n_1^d n_2^c$ is the binormal to the bifurcation surface $\tilde{B}$ respectively.

### 7.3.1 For $L(g^{ab}, R_{cdef})$

Evaluated on an AdS background where $R_{abcd} = -(g_{ac}g_{bd} - g_{ad}g_{bc})$, it can be shown (see appendix D.1) that the Wald functional and its linear variation for (7.7) (without covariant derivatives of curvature terms in the action) takes the simple form

$$E^{abcd}_R = c_1 g^{(ab} g^{cd)},$$

(7.41)

and,

$$\delta E^{abcd}_R = -c_2 g^{(ab} g^{cd)} h - c_3 h^{(ab} g^{cd)} + c_4 g^{(ab} g^{cd)} R + c_5 R^{(ab} g^{cd)} + c_6 R^{abcd},$$

(7.42)

where all the coefficients are not independent but related by [5]

$$c_2 = -2d c_4 - c_5, \quad c_3 = 2c_1 - (d - 1)c_5 - 4c_6,$$

(7.43)

and $\langle \cdot \rangle$ implies that it has been properly (anti)symmetrized to have the properties of the Riemann tensor. The linearized Reimann tensor is given by

$$R_{abcd} = \frac{1}{2} (\nabla_c \nabla_b h_{ad} - \nabla_d \nabla_b h_{ac} + \nabla_d \nabla_a h_{bc} - \nabla_c \nabla_a h_{bd}) + \frac{1}{2} (R_{acbd} h^c_b + R^r_{bd} h_{ac}).$$

(7.44)
When $\delta T_{\mu\nu}$ is small and $R \to 0$, the scaling analysis in [5] shows that at the leading order we can neglect all the derivatives $\partial_\mu \neq z$ in comparison to $\partial_z$. Near the boundary, the metric perturbations can be written as

$$h_{\mu\nu}(z, x^\lambda) = z^{\Delta - 2} h_{\mu\nu}(x^\lambda) + \ldots$$  \hspace{2cm} (7.45)

Using (7.44) the relevant components of the linearized Wald functional in (7.2) take the form

$$\delta E^{(1)\mu z\nu z}_R = A h_{\mu\nu}^z g_{zz} + B h_{\mu\nu} g_{zz}^z, \quad \delta E^{(1)\mu\nu\rho\sigma}_R = C h_{\mu\nu}^\rho g_{\rho\sigma} + D h_{\mu\nu} (g_{\rho\sigma}),$$  \hspace{2cm} (7.46)

where the coefficients $A, B, C, D$ are functions of the coefficients $c_1 \ldots c_6$ in (7.7). Substituting (7.46) and (7.41) into (7.40), we get,

$$\delta S_{Wald} = 4\pi \tilde{L}^{d-3} \int_A \frac{d^{d-1}x}{z^{d-2}} (A_1 h_{tt} + A_2 h_{\mu\nu}^\rho h_{\mu\nu}),$$  \hspace{2cm} (7.47)

After putting $\Delta = d$ in order to get a finite answer [5] we find

$$A_1 = 2a - \frac{D}{2} (d - 2) R^2 + \left[ c_1 \frac{D}{2} (d - 2) + 2A(d - 1) \right] \left[ \frac{|x|^2}{d - 1} - R^2 \right],$$

$$A_2 = \left[ c_1 \frac{D}{2} (d + 1) + (C - 2B)(d - 1) \right] \left[ \frac{|x|^2}{d - 1} \right].$$  \hspace{2cm} (7.48)

Performing the integral in (7.47) and using (7.38) we have,

$$\delta T_{\mu\nu}^{grav} = \alpha h_{tt}^{(d)} + \beta \eta_{\mu\nu} h_{(d)}^\mu,$$  \hspace{2cm} (7.49)

where the coefficients are given as

$$\alpha = d(-c_1 + c_3 + (d - 1)c_5 + 2dc_6),$$

$$\beta = \left[ -(d + 2)c_1 + 2(d + 1)c_2 + c_3 + 2d(d + 1)c_4 + (d + 1)c_5 - 2(d - 2)c_6 \right],$$  \hspace{2cm} (7.50)

and $h_{(d)}^\mu$ has no $z$ dependence. These can be generalized to an arbitrary Lorentz frame and combined with the tracelessness and conservation equations $h_{(d)}^{(d)\mu} = 0$, $\partial_\mu h_{(d)\mu} = 0$ we have (7.5) as

$$\delta T_{\mu\nu}^{grav} = d\tilde{L}^{d-3} [c_1 + 2(d - 2)c_6] h_{(d)}^\mu.$$  \hspace{2cm} (7.51)

### 7.3.2 For $\mathcal{L}(g^{ab}, R_{cdef}, \nabla_a R_{bcde})$

The above analysis can be extended to actions containing covariant derivatives on the Riemann tensors. The most general term containing arbitrary covariant derivatives on the curvature tensors is deferred for future work; we consider here $\mathcal{L}(g^{ab}, R_{cdef}, \nabla_a R_{bcde})$. The background

\footnote{The explicit dependencies of $A, B, C$ and $D$ on the coefficients $c_1 \ldots c_6$ are given in footnote 20 of [5].}
field expansion of the action at $O((\Delta R)^2)$ is given by

$$S_{\nabla R} = \int d^{d+1}x \sqrt{g} Z^{abcdmnrs} \nabla_e \Delta R_{abcd} \nabla_f \Delta R_{mnrs}. \quad (7.52)$$

It can be shown (see appendix B) that the above action can be written as

$$S = \int d^{d+1}x \sqrt{g} [d_1 \Delta R_{abcd} \nabla^2 \Delta R_{abcd} + d_2 \Delta R_{ab} \nabla^2 \Delta R_{ab} + d_3 \Delta R \nabla^2 \Delta R]. \quad (7.53)$$

Since $\nabla_a g_{bc} = 0$, we can write

$$S = \int d^{d+1}x \sqrt{g} [d_1 \Delta R_{abcd} \nabla^2 R_{abcd} + d_2 \Delta R_{ab} \nabla^2 R_{ab} + d_3 \Delta R \nabla^2 R]. \quad (7.54)$$

At the linear order in fluctuations using (7.44)

$$R_{L}^{ab} = \frac{1}{2} \nabla^c \nabla_a h_{bc} + \frac{1}{2} \nabla_b \nabla^d h_{ad} - \frac{1}{2} \nabla^2 h_{ab} - \frac{1}{2} \nabla_a \nabla_b h + \frac{1}{2} (R_{abcd} h^{cd} + R_{ab} h_{cd}),$$

$$R_{L} = \nabla^a \nabla_b h_{ab} - \nabla^2 h - dh. \quad (7.55)$$

If we consider the transverse, traceless gauge $\nabla_a h_{ab} = 0, h^a_a = 0$, we have,

$$R_{L} = 0, \text{ and } R_{ab}^L = -[\frac{1}{2} \Box + d] h_{ab}. \quad (7.56)$$

We can see that $\nabla^2 R$ term will not contribute to the action. To see that $(\nabla_a \Delta R_{bc})^2$ will also not contribute to the action, we will first carry out the linearization of $\Delta R_{ab}$ which is given by

$$\Delta R_{abcd}^{L} = R_{abcd}^{L} - \bar{R}_{abcd}^{L} = R_{abcd}^{L} + (g^{(0)}_{ac} h_{bd} + g^{(0)}_{bd} h_{ac} - g^{(0)}_{ad} h_{bc} - g^{(0)}_{bc} h_{ad}). \quad (7.57)$$

Contracting with $g^{(0)}_{bd}$ we get,

$$\Delta R_{ac}^{L} = R_{ac}^{L} + (d - 1) h_{ac} = -\frac{1}{2} [\Box + 2] h_{ac}. \quad (7.58)$$

This term vanishes on using the lowest order equation of motion for $h_{ab}$. Thus this term does not contribute to the holographic stress tensor. For the remaining $\nabla_a R_{bcde}$ terms we can use the Bianchi Identity as in [15] to put the final expression [see appendix (D.2)] in the form (neglecting the total derivatives),

$$R_{bcde} \nabla^2 R_{bcde} = -4(\nabla_d R_{ce})^2 - (\nabla R)^2 - 4 R_{cd}^{ae} R_{d}^{e f} R_{e f} - 4 R_{d}^{ae} R_{f}^{ed} R_{e}^{d f} + 2 R_{bcde}^{l} R_{l}^{f} R_{f}^{cde} + 2 R_{bcde}^{a} R_{a}^{f} R_{f}^{bc} + 2 R_{bcde}^{e} R_{e}^{f} R_{f}^{bc} + 2 R_{bcde}^{f} R_{f}^{e} R_{e}^{bc} + 2 R_{bcde}^{a} R_{a}^{f} R_{f}^{bc} + 2 R_{bcde}^{b} R_{b}^{f} R_{f}^{bc} R_{acfe}. \quad (7.59)$$

The $O(R^3)$ terms in the expression are given by

$$S_{R^3} = -4 R_{dc}^{cd} R_{e f}^{d e} R_{e f}^{d e} + 2 R_{bcde}^{l} R_{l}^{f} R_{f}^{cde} + 2 R_{bcde}^{a} R_{a}^{f} R_{f}^{bc} + 4 R_{bcde}^{a} R_{bcde}^{a} f R_{acfe}. \quad (7.60)$$
Doing a similar background field expansion of the above terms we have at the second order in the expansion,

$$O(\Delta R^2) = 4(d+2)\Delta R^{ab}\Delta R_{ab} - 4(\Delta R)^2 - 2d\Delta R^{abcd}\Delta R_{abcd},$$

(7.61)

and at the first order there is no contribution $$O(\Delta R) = 0.$$ Hence, the coefficients that get shifted are

$$c'_4 = c_4 - 8d, \quad c'_5 = c_5 + 8(d+2)d, \quad c'_6 = c_6 - 4d d,$$

(7.62)

while $$c_1$$ remains unchanged. Putting these values in the expression for $$\delta T_{\mu\nu}^{grav},$$ we have,

$$\delta T_{\mu\nu}^{grav} = d\tilde{L}d^{-3}[c'_4 + 2(d-2)c'_6]h^{(d)}_{\mu\nu} = d\tilde{L}d^{-3}[c_1 + 2(d-2)(c_6 - 4d d)]h^{(d)}_{\mu\nu}.$$ 

(7.63)

We can also calculate the holographic stress tensor in (7.5) directly (see appendix (D.3)) and show the shift in the coefficient $$c_6$$ explicitly.

### 7.4 Holographic trace anomalies

#### 7.4.1 For $$\mathcal{L}(g^{ab}, R_{cdef})$$

We will now calculate the holographic trace anomalies [20, 21] for the Lagrangian in (7.7) following a simple method advocated in Appendix A of [22]. This method can be easily implemented on a computer. Our results will be in agreement with [8] wherever we have been able to compare our expressions. We outline the essential steps in the computation of the anomalies.

1. We will first choose a reference background for $$g^{(0)}_{ij}$$. Since there is no restriction, we can choose any reference background, convenient for the calculation. Note that we can also use multiple reference background for $$g^{(0)}$$ to determine all the anomaly coefficients.

2. The form of $$g^{(1)}_{ij}$$ is fixed by conformal invariance as [23]

$$g^{(1)}_{ij} = -\frac{1}{d-2}(R^{(0)}_{ij} - \frac{R^{(0)}}{2(d-1)}g^{(0)}_{ij}),$$

(7.64)

where $$R^{(0)}$$s are constructed out of $$g^{(0)}$$ respectively.

3. We will keep $$g^{(2)}_{ij}$$ arbitrary. Some comments are in order. Demanding the coefficient to $$g^{(2)}$$ to vanish in $$d = 4$$ in the Lagrangian enforces the condition $$c_0 = -8c_1.$$ This is the same condition as obtained from the lowest order equations of motion. For $$d = 6$$ the relation between $$c_0$$ and $$c_1$$ is obtained by demanding that the coefficient of $$g^{(3)}_{ij}$$ vanishes. We put in $$g^{(2)}_{ij}$$ for consistency but in the end it does not play a role.

4. Plugging in the FG expansion in (7.6) into (7.7), we get

$$S = \int dz^d x z^{-d-1} \sqrt{-g^{(0)}} b(x, z),$$

(7.65)
where \( b(x, z) = b_0(x) + z^2 b_1(x) + \ldots \). Next we extract the coefficient of \( 1/z \) term in the above term which we call \( S_{ln} \).

5. The trace anomaly in \( d \) dimensions is given by

\[
\langle T_{\mu}^{\mu} \rangle = b_{d/2}, \tag{7.66}
\]

where \( b_{d/2} \) is the coefficient of \( z^d \) in the expansion for \( b(x, z) \).

6. By matching the term \( S_{ln} \) with the expressions for \( \langle T_{\mu}^{\mu} \rangle \) we can determine various anomaly coefficients.

### 7.4.1.1 \( d=2 \)

In \( d=2 \) the \( S_{ln} \) has only one anomaly term which is the Euler anomaly given by \( E_2 = \frac{1}{4} R \).

Evaluated on the manifold

\[
ds^2 = g_{(0)ij} dx^i dx^j = u(\chi^2 dt^2 + \frac{d\chi^2}{\chi^2}), \tag{7.67}
\]

the Euler anomaly takes the form \( E_2 = -\frac{1}{2u} \). The \( 1/z \) term in the action is given by \( S_{ln} = -2c_1 \).

Equating this with the anomaly term \( \mathcal{A} = \frac{c}{2\pi} E_2 \) and finally putting \( u = 1 \), we get

\[
c = 32\pi \tilde{L} c_1. \tag{7.68}
\]

### 7.4.1.2 \( d=4 \)

In \( d=4 \) the \( S_{ln} \) will contain a linear combination of the Weyl and the Euler anomalies given by

\[
E_4 = R_{(0)}^{abcd} R_{(0)abcd} - 4 R_{(0)}^{ab} R_{(0)ab} + R_{(0)}^2, \\
I_4 = R_{(0)}^{abcd} R_{(0)abcd} - 2 R_{(0)}^{ab} R_{(0)ab} + \frac{1}{3} R_{(0)}^2, \tag{7.69}
\]

where \( R_{(0)abcd} \) is constructed out of \( g_{(0)ab} \). The trace anomaly is given by

\[
\langle T_{\mu}^{\mu} \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4. \tag{7.70}
\]

We take \( g_{(0)} \) as

\[
g_{(0)i,j} dx^i dx^j = u(-\chi^2 dt^2 + \frac{d\chi^2}{\chi^2}) + v(d\theta^2 + \sin^2 \theta d\phi^2), \tag{7.71}
\]

which is of the form \( AdS_2 \times S_2 \). In this background the anomalies take the form

\[
E_4 = -\frac{8}{uv}, \quad I_4 = \frac{4(u - v)^2}{3u^2v^2}. \tag{7.72}
\]
Chapter 7. Holographic stress tensor at finite coupling

The coefficient of $1/z$ term in the action is

$$S_{ln} = \frac{\sin \theta}{6uv}(4c_6(u-v)^2 + c_1(u^2 + 4uv + v^2)).$$  (7.73)

Comparing $S_{ln}$ and $\langle T^\mu_\mu \rangle$ we get, after restoring the factors of $\tilde{L}$ in $a$ and $c$

$$a = 2\pi^2 \tilde{L}^3 c_1, \quad c = 2\pi^2 \tilde{L}^3 (c_1 + 4c_6).$$  (7.74)

The 4d holographic stress tensor in (7.5) can thus be written as

$$\langle \delta T^{grav}_{\mu\nu} \rangle = 4\tilde{L}[c_1 + 4c_6]h^{(d)}_{\mu\nu} = \frac{2}{L^2\pi^2} c h^{(d)}_{\mu\nu}.$$  (7.75)

7.4.1.3 $d=6$

In $d=6$ there are four anomaly coefficients [24, 25] of which three are called the B-type anomalies which are the coefficients of the three Weyl anomalies and the other one is the A-type which is the coefficient of the Euler term in 6d. The trace anomaly in 6d is given by

$$\langle T^\mu_\mu \rangle = S_{ln} = (\sum_{i=1}^{3} B_i I_i + 2AE_6),$$  (7.76)

where the expressions for the anomalies are given by

$$I_1 = C_{ijkl}C^{lmij}C_{mn}^{\phantom{mn}kl},
I_2 = C_{ij}^{\phantom{ij}kl}C^k_lC_{mn}^{\phantom{mn}ij},
I_3 = C_{ijkl}(\nabla^2\delta_j^i + 4R_j^i - \frac{6}{5}R\delta_j^i)C_{jklm},$$  (7.77)

$$E_0 = 384\pi^3 E_6 = K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8,$$

where the terms $K_1 \ldots K_8$ are given by (7.8). To determine the anomaly coefficients we choose for $g_{(0)}$ two manifolds $AdS_2 \times S_4$ and $AdS_2 \times S_2 \times S_2$. In $AdS_2 \times S_4$ we have

$$I_1 = -\frac{51(u-v)^3}{100u^4v^3}, \quad I_2 = \frac{39(u-v)^3}{25u^4v^3}, \quad I_3 = -\frac{36(19u+v)(u-v)^2}{25u^4v^3}, \quad E_6 = -\frac{144}{uv^2},$$  (7.78)

while in the $AdS_2 \times S_2 \times S_2$ background we have

$$I_1 = -\frac{3(51u^3 + 21u^2v + 17uv^2 - 17v^3)}{100u^4v^3}, \quad I_2 = \frac{3(39u^3 - 31u^2v + 13uv^2 + 13v^3)}{25u^3v^3},$$
$$I_3 = -\frac{12(11u^3 - 39u^2v + 17uv^2 + 3v^3)}{25u^3v^3}, \quad E_6 = -\frac{48}{uv^2}.$$  (7.79)

$S_{ln}$ in the $AdS_2 \times S_3$ background takes the form,

$$S_{ln} = -\frac{3c_1}{4} - (c_1 + 8c_6)\frac{3(u-v)^2}{20v^2} + (11c_1 + 94c_6 + 104\tilde{c}_1 - 34\tilde{c}_8)\frac{3(u-v)^3}{200v^3},$$  (7.80)
where $\tilde{c}_7$ and $\tilde{c}_8$ are coefficients of the seventh and the eighth term in (7.8). Comparing (7.80) and (7.76), we get

$$A = c_1, \quad B_3 = \frac{1}{192}(8c_6 + c_1), \quad 68B_1 - 208B_2 + c_1 + 20c_6 + 208\tilde{c}_7 - 68\tilde{c}_8 = 0. \quad (7.81)$$

Using $AdS_2 \times S_2 \times S_2$ for $g(0)$ we get one more relation as,

$$54B_1 - 24B_2 + 3c_1 + 10c_6 + 24\tilde{c}_7 - 54\tilde{c}_8 = 0. \quad (7.82)$$

We solve these two equations to get after restoring the factors of $\tilde{L}$,

$$A = \tilde{L}^5c_1, \quad B_3 = \frac{\tilde{L}^5}{192}(8c_6 + c_1), \quad 2B_1 = \tilde{L}^5(- \frac{c_1}{8} - \frac{c_6}{3} + 2\tilde{c}_8), \quad 2B_2 = \tilde{L}^5(- \frac{c_1}{32} + \frac{c_6}{12} + 2\tilde{c}_7). \quad (7.83)$$

The holographic stress tensor in (7.5) can now be re-expressed as,

$$\langle \delta T^\text{grav}_{\mu\nu} \rangle = 6\tilde{L}^3[c_1 + 8c_6]h^{(6)}_{\mu\nu} = 6\tilde{L}^3B'_3h^{(6)}_{\mu\nu}, \quad (7.84)$$

where we define $B'_3 = 192B_3$. The relation between the holographic stress tensor and the asymptotic metric thus takes the form of (7.5) where $C_T$ is related to the B-type anomaly coefficient as

$$C_T = 192f_6B_3. \quad (7.85)$$

7.4.2 For $L(g^{ab}, R_{cdef}, \nabla_a R_{bced})$

We will use the same Lagrangian (7.54) for the calculation of the holographic anomalies. Here by a scaling argument as in [8] it is easy to show that the action with two covariant derivatives acting on two Riemann tensors, will take on the form as in (7.54). In the presence of the $\nabla R$ terms in the action, the central charges of the higher derivative theories get modified accordingly.

7.4.2.1 d=4

The additional contribution to the $S_{ln}$ is

$$S_{(d)ln} = -\frac{32d_3\sin \theta (u - v)^2}{3uv}, \quad (7.86)$$

which combined with the remaining terms give

$$S_{ln} = \frac{\sin \theta}{6uv} [(4c_6 - 64d_3)(u - v)^2 + c_1(u^2 + 4uv + v^2)]. \quad (7.87)$$

\textsuperscript{7}We thank Mark Mezei for point out a mistake in the previous version. The difference arose from missing out a factor of 3 in(7.80)
Comparing these expressions with the usual formula for the anomaly term we get the anomaly coefficients, as
\[ a = 2\pi^2 \tilde{L}^3 c_1, \quad c = 2\pi^2 \tilde{L}^3 (4c_6 + c_1 - 64d_3). \] (7.88)

We can say that \( c'_6 = c_6 - 16d_3 \) and hence the holographic stress tensor of (7.5) becomes
\[ \langle \delta T_{\mu\nu}^{\text{grav}} \rangle = 4\tilde{L}[c_1 + 4c'_6]h^{(4)}_{\mu\nu} = \frac{2}{L^2\pi^2} c h^{(4)}_{\mu\nu}, \] (7.89)
as before for \( 4d \).

### 7.4.2.2 \( d=6 \)

In \( 6d \) the additional contribution to \( S_{\text{ln}} \) due to the \( (\nabla R)^2 \) terms in \( \text{AdS}_2 \times \text{S}_4 \) is,
\[ S_{(d)\text{ln}} = -\frac{36(7u + 3v)(u - v)^2}{25u^2v} d_3. \] (7.90)

Comparing the total contribution to the coefficient of \( 1/z \) term with the expression for \( \langle T_{\mu\nu}^a \rangle \) for \( \text{AdS}_2 \times \text{S}_4 \) and \( \text{AdS}_2 \times \text{S}_2 \times \text{S}_2 \) we get, after restoring the factors of \( \tilde{L} \),
\[ A = \tilde{L}^5 c_1, \quad B_3 = \frac{\tilde{L}^5}{192} (c_1 + 8c_6 - 192d_3), \quad 2B_1 = \tilde{L}^5 \left(-\frac{c_1}{8} - \frac{c_6}{3} + 2c_8 + 16d_3\right), \quad 2B_2 = \tilde{L}^5 \left(-\frac{c_1}{32} + \frac{c_6}{12} + 2c_7 - 4d_3\right) \] (7.91)
where \( c'_6 = c_6 - 4dd_3 \). The holographic stress tensor in (7.5) can now be written as
\[ \delta T_{\mu\nu}^{\text{grav}} = 6\tilde{L}^3[c_1 + 8c'_6]h^{(6)}_{\mu\nu} = 6\tilde{L}^3 B'_3 h^{(6)}_{\mu\nu}, \] (7.92)
for the \( 6d \) case where as before we define \( B'_3 = 192B_3 \).

### 7.5 Holographic two point function for higher derivative theories in arbitrary dimensions

In this section we will show that the coefficient in the expression for the holographic stress tensor is related to the coefficient in the holographic two point function in arbitrary dimensions for any higher derivative theory whose bulk Lagrangian is of the form \( \mathcal{L}(g^{ab}, R_{bcde}, \nabla_a R_{bcde}) \). In even dimensions the coefficient of the holographic two point function is related to the coefficient of the two point function in field theory which is proportional to the B-type anomaly coefficient [13], [14] (our results in six dimensions are new). The details of the calculation from the field theory side are done in appendix (D.4). In odd dimensions there is no anomaly. We will show that the coefficient appearing in the expression of the holographic stress tensor is related to the coefficient of the holographic two point functions in arbitrary dimensions.
As previously, we will consider the action,

$$S = \int d^{d+1}x \sqrt{|g|} c_0 + c_1 \Delta R + \frac{c_4}{2} \Delta R^2 + \frac{c_5}{2} \Delta R^{ab} \Delta R_{ab} + \frac{c_6}{2} \Delta R^{abcd} \Delta R_{abcd}, \quad (7.93)$$

where $c_0 = -2dc_1$. The advantage of using the above action for the computation of the two point function is that the result is then valid for any arbitrary higher derivative theory of gravity of the form $L(g^{ab}, R_{bcde}, \nabla_a R_{bcde})$ with $c_6$ replaced by $c'_6$ as argued previously. To compute the two point function it is sufficient to keep up to $O(\Delta R)^2$ terms only since as we are expanding when we expand around the AdS background, $O(\Delta R)^3$ terms will start at order $O(h^3)$. To compute the two point functions we will follow the arguments of [12] where it is shown that to calculate the two point functions it is sufficient to look at components like $\langle T_{xy} T_{xy} \rangle$ since the other structures are completely determined by symmetry. Following [12] we turn on a component $r^2h_{xy}(r, z)/L^2$ of the metric perturbations. The quadratic action for the fluctuation of the above form for our case is given by

$$S = \int d^{d+1}x \{K_1 \dot{\phi}^2 + K_2 (\partial_r \phi)^2 + K_3 \partial_z \partial_z \phi^2 + K_4 \partial_z \phi \partial_r \phi + K_5 (\partial_r \phi)^2 + K_6 (\partial_r \partial_z \phi)^2 + K_7 \partial_z \phi \partial_z \phi + K_8 \partial_r \phi \partial_r \phi + K_9 (\partial_r \phi)^2 + K_{10} \partial_z \phi \partial_z \phi + K_{11} \partial_r \phi \partial_z \phi + K_{12} \partial_r \partial_z \phi \partial_r \partial_z \phi \} \quad (7.94)$$

The last term can be integrated by parts to obtain

$$K_{12} \phi \partial_r ^2 \phi = \partial_r (K_{12} \phi \partial_r \phi) - K_{12} (\partial_r \phi)^2 - \partial_r K_{12} \phi \partial_r \phi, \quad (7.95)$$

where we have assumed that there exists a generalized Gibbons-Hawking term which takes care of the total derivatives. We will consider the scalar field to be

$$\phi(r, z) = \phi_k(r)e^{-ikz}. \quad (7.96)$$

Taking the Fourier transform of the action, after the integration by parts of the last term, we have

$$A = \int d^{d+1}k \{K_1 \phi_k \phi_{-k} + K_2 k^2 \phi_k \phi_{-k} + K_3 k 4 \phi_k \phi_{-k} - \frac{1}{2} K_4 k^2 \phi_k \phi_{-k} - \frac{1}{2} K_4 k^2 \phi_{-k} \phi_k + K_5 \phi_k \phi_{-k} + K_6 \phi_k \phi_{-k} + K_7 k^2 \phi_k \phi_{-k} + \frac{1}{2} \partial_r (K_7 k^2 \phi_k) \phi_{-k} - \frac{1}{2} \partial_r (K_7 k^2 \phi_{-k}) \phi_k - \frac{1}{2} K_8 \phi_k \phi_{-k} - \partial_r (K_9 \phi_k) \phi_{-k} + K_{10} k^2 \phi_k \phi_{-k} + \frac{1}{2} K_{11} \phi_k \phi_{-k} + K_{12} \phi_k \phi_{-k} - K_{12} \phi_{-k} \phi_k \} \quad (7.97)$$
where denotes derivative with respect to \( r \). The terms \( K_i \) are given by

\[
\begin{align*}
K_1 &= dc_1r^{d-1}, \\
K_2 &= \frac{3}{2}c_1r^{d-3}, \\
K_3 &= \frac{C_5}{4} + c_6, \\
K_4 &= \frac{1}{2}(d+1)c_3 + 4c_6) \right] r^{d-2}, \\
K_5 &= \frac{1}{4}[(d+1)^2 c_5 + 4(d+7)c_6] r^{d+1}, \\
K_6 &= 2c_6 r^{d-1}, \\
K_7 &= \frac{1}{2} c_5 r^{d-1}, \\
K_8 &= \frac{1}{2}[(d+1)c_5 + 12c_6] r^{d+2}, \\
K_9 &= \frac{1}{4}(c_5 + 4c_6) r^{d+3}, \\
K_{10} &= 2c_1r^{d-3}, \\
K_{11} &= 2(d+2)c_1 r^d, \\
K_{12} &= 2c_1 r^{d+1}.
\end{align*}
\]

(7.98)

After integration by parts the above action can be written as a boundary term \(^8\)

\[
\partial \mathcal{A} = \frac{1}{2}K_kk^2 \phi_k \phi_{-k} + K_5 \phi_k \phi_{-k} + K_6 \phi^2_k \phi_{-k} + \frac{1}{2} \partial_r(K_7k^2 \phi_k) \phi_{-k} - \frac{1}{2} K_8 \phi_k \phi_{-k} - \partial_r(K_9 \phi_k) \phi_{-k} - K_{12} \phi_k \phi_{-k},
\]

(7.99)

where again denotes derivative with respect to \( r \). The solution to \((\Box + 2) h_{ab} = 0\) still solves the higher derivative equations\(^9\). The solution is given by (restoring the AdS radius \( \tilde{L} \))

\[
\phi_k(r) = \frac{2 \tilde{L}^4 |k|^{d/2}}{d^{d/2} \Gamma[d/2]} K_{d/2}(\frac{\tilde{L}^2 |k|}{r}),
\]

(7.100)

where \( K_{d/2} \) is the modified Bessel function of the second kind. The normalization constant is obtained by imposing the condition that \( \phi_k(r = \infty) = 1 \) and \( d \) is the field theory dimension. By plugging this solution back into the surface term \( \partial \mathcal{A} \) and extracting the coefficient of \( k^d \) term in the resulting expression, we get, for \( AdS_{d+1}/CFT_d \) after restoring the factors of \( \ell_p \),

\[
\langle T_{ab}(x) T_{cd}(x') \rangle = \frac{1}{\tilde{L}^2 C_T} \frac{\mathcal{I}_{ab,cd}(x-x')}{|x-x'|^{2d}},
\]

(7.101)

where the coefficient \( C_T \) is given by

\[
C_T = f_d \tilde{L}^d [c_1 + 2(d-2)c_6],
\]

(7.102)

where \( f_d \) is the constant factor given by

\[
f_d = 2 \frac{d+1}{d-1} \frac{\Gamma[d+1]}{\pi^{d/2} \Gamma[d/2]},
\]

(7.103)

Thus the expression for \( \langle T^{grav}_{\mu \nu} \rangle \) in (7.5) becomes,

\[
\langle T^{grav}_{\mu \nu} \rangle = \frac{d}{f_d \tilde{L}^2} C_T h^{(d)}_{\mu \nu}.
\]

(7.104)

Note that while we have assumed the existence of a suitable generalized Gibbons-Hawking term we have not used counter terms involving boundary curvature tensors in our calculations. We

\(^8\)We have assumed that the volume counter term gets rid of \( \phi_k \phi_{-k} \) terms as in \([9, 27]\).

\(^9\)See e.g.\([26]\), alternatively we just assume that there is a massless graviton which by definition solves this equation.
have explicitly checked, the addition of such counter terms will not alter our findings.

### 7.6 Three point functions

The fact that we were able to get the one point and two point functions from the background field expansion seems to suggest that the analysis can be extended to the calculation of 3-point functions using the same technique. We will carry out the analysis first by considering a higher derivative Lagrangian of the form $\mathcal{L}(g^{ab}, \Delta R_{abcd})$ and then extending the analysis to the case where $\mathcal{L}(g^{ab}, \Delta R_{abcd}, \nabla_a \Delta R_{bcde})$.

Direct holographic calculation of the three point functions are involved and challenging. We will follow the alternative route to derivation of the three point functions following the analysis of [10] and used in [10, 12, 28]. The energy flux associated with a localized perturbation of fixed energy $\epsilon_{ij} T^{ij}$, where $\epsilon_{ij}$ is the polarization tensor, in the $d (> 3)$ dimensional CFT background is given by

$$
\langle \epsilon(n) \rangle = \frac{E}{4\pi \Omega_{d-2}} [1 + t_2 (\epsilon^*_{ij} \epsilon_{kj} n^k - \frac{1}{d-1}) + t_4 (\frac{\epsilon_{ij} n^j n^k}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{2}{d^2 - 1})].
$$

(7.105)

Here $E$ is the total energy flux, $n$ is the outward normal in the direction in which the flux is measured and $\Omega_{d-2}$ is the volume of a unit $S^{d-1}$ sphere. The coefficients $t_2$ and $t_4$ are determined holographically in the following way. From the holographic side we insert graviton perturbations $h_{\mu\nu}$ dual to the energy insertion in the field theory and evaluate the on-shell cubic term in the higher derivative Lagrangian corresponding to these graviton insertions. Following [10, 12], we consider the shock wave background with perturbations in $d$ dimensions:

$$
ds_{sw}^2 = \frac{\tilde{L}^2}{u^2} \left[ \delta(y^+) W(\tilde{y}, u)(dy^+)^2 - dy^+ dy^- + d\tilde{y}^2 + du^2 \right] + h_{ij} dx^i dx^j,
$$

(7.106)

where $\tilde{y}^2 = \sum_{i=0}^{d-2} y_i^2$ and $d$ is the dimension of the field theory. The function $W(\tilde{y}, u)$ is given by

$$
W(\tilde{y}, u) = \frac{2^{d-1}}{(1 + n_{d-1})^{d-1}} \frac{u^d}{(u^2 + (\tilde{y}^2 - \tilde{Y}^2)^2)^{d-1}},
$$

(7.107)

where $n_{d-1}$ is the $(d - 1)$th component of the normal vector given by

$$
n_{d-1} = (1 - n_i^2)^{\frac{1}{2}}, \quad \text{and} \quad \tilde{Y}^i = \frac{n_i}{1 + n_{d-1}}.
$$

(7.108)

$W$ satisfies the following equation in any higher derivative theory of gravity[29],

$$
\partial_y^2 W - \frac{d - 1}{u} \partial_u W + \sum_{i=1}^{d-2} \partial_{y^i}^2 W = 0.
$$

(7.109)

The transverse traceless gauge brings down the number of independent components of the perturbations. In $d$ dimensions we can consider the perturbation of the form $h_{y^1 y^2} = \tilde{L}^2 / u^2 \phi(\tilde{y}, u)$,
while \( h = 0 = \nabla^{\mu}h_{\mu\nu} \) relates the other components as
\[
\partial_- h_{y^+y^1} = \frac{1}{2} \partial_{y^2} h_{y^1y^2}, \quad \partial_- h_{y^1y^2} = \frac{1}{2} \partial_{y^1} h_{y^1y^2}, \quad \partial_- h_{y^2y^1} = \frac{1}{4}(\partial_{y^1} h_{y^1y^2} + \partial_{y^2} h_{y^1y^2}).
\] (7.110)

It is sufficient to turn on these components only for general \( d(> 3) \) dimensions. The component \( h_{y^1y^2} \) satisfies the lowest order equation of motion for a scalar field in the \( AdS_{d+1} \) background given by,
\[
\partial_y^2 \phi - \frac{d-1}{u} \partial_u \phi + \sum_{i=0}^{d-2} \partial_{y^i}^2 \phi - 4 \partial_+ \partial_- \phi = 0.
\] (7.111)

### 7.6.1 \( \mathcal{L}(g^{ab}, R_{cdef}) \)

Using the equation of motion for \( \phi \) and \( W \) we can evaluate the on-shell cubic effective action to get the most general form in \( d(> 3) \) dimensions as\(^{10}\)
\[
S_{W,\phi}^{(3)} = -\frac{1}{4} \int d^{d+1}x \sqrt{-g} \phi \partial_\mu \phi \left[ 2(c_1 + 2(d-2)c_6) W - 2u(2c_6 - 12d \phi_7 + 3(3d - 4) \phi_8) \partial_u W \right.
\]
\[
- 24u^2(2 \phi_7 - \phi_8) \sum_{i>2} \partial_{y^i}^2 W + u^2(2c_6 - 12(8-d) \phi_7 + 3(12 - d) \phi_8)(\partial_{y^1}^2 W + \partial_{y^2}^2 W)
\]
\[
- 24u^3(2 \phi_7 - \phi_8)(\sum_{i=1}^{d-1} \partial_{y^i}^2 \partial_u W - u \sum_{i>j} \partial_{y^i}^2 \partial_{y^j} W) \mid \substack{u=1, y_1=0, y_2=0 \end{align}
\] (7.112)

Note that the integral localizes on \( u = 1, y_1 = 0, y_2 = 0 \) \([10, 11, 12]\). As a result we do not have to worry about boundary terms like the generalized Gibbons-Hawking term or the boundary counter terms in this calculation. Comparing with the standard form given in \([12]\),
\[
S_{W,\phi}^{(3)} = -\frac{C_T}{4 f_
u L_{d-1}} \int d^{d+1}x \sqrt{-g} \phi \partial_\mu \phi \ W [1 + t_2 T_2 + t_4 T_4],
\] (7.113)

and \( T_2 \) and \( T_4 \) are given by
\[
T_2 = \frac{n_1^2 + n_2^2}{2} - \frac{1}{d-1}, \quad T_4 = 2n_1^2 n_2^2 - \frac{2}{a^2 - 1},
\] (7.114)

while the coefficients \( t_2, t_4 \) are given by\(^{11}\),
\[
t_2 = \frac{d(d-1)}{c_1 + 2(d-2)c_6}[2c_6 - 12(3d+4) \phi_7 + 3(7d+4) \phi_8], \quad t_4 = \frac{6d(d^2 - 1)(d + 2)}{c_1 + 2(d - 2)c_6}(2 \phi_7 - \phi_8).
\] (7.115)

This is the expected result for cubic Lovelock theory \([28]\) where \( 2 \phi_7 = \phi_8 \) and hence \( t_4 = 0 \). We have also checked that our general expressions are in agreement with \([12, 30]\).

---

\(^{10}\)To reach this simple form, we need to integrate by parts and use the on-shell conditions multiple number of times.

\(^{11}\)If we set \( W = 1 \) then we would be left with just the two point function which would be proportional to \( C_T \) as expected.
7.6.2 \( \mathcal{L}(g^{ab}, R_{cdef}, \nabla_a R_{bcde}) \)

We now extend the analysis of the previous section to higher curvature theories containing covariant derivatives of the Riemann tensor. In section (7.3.2) we have shown how the presence of the \( \nabla^2 \Delta R^2 \) terms modify the coefficient \( c_6 \rightarrow c'_6 = c_6 - 4dd_3 \). In addition the cubic order coefficients are modified as

\[
\tilde{c}_7 \rightarrow \tilde{c}_7' = \tilde{c}_7 - 3d_3 \tilde{c}_8 = \tilde{c}_8 + 4d_3, \quad \tilde{c}_2 \rightarrow \tilde{c}_2' = \tilde{c}_2 - 4d_3, \quad \tilde{c}_5 \rightarrow \tilde{c}_5' = \tilde{c}_5 + 4d_3, \quad \tilde{c}_6 \rightarrow \tilde{c}_6' = \tilde{c}_6 + 2d_3.
\]

(7.116)

Thus \( c_i \) and \( \tilde{c}_i \) in (7.115) will be replaced by \( c'_i \) and \( \tilde{c}'_i \) respectively. In this section we will consider additional terms like \( \nabla^2 \Delta R^3 \) terms in the action (7.7). For \( \nabla \nabla \Delta R^3 \) terms, since the linearized Ricci tensor and scalar curvature vanishes by using the tracelessness condition and the lowest order equation of motion satisfied by \( h_{ab} \), as shown in section (7.3.2), the terms which contribute to the three point functions are

\[
S_3 = e_1 R^{ab}_{\quad cd} R^{cd}_{\quad ef} \nabla^2 R^{ef}_{\quad ab} + e_2 R^a_{\quad c} R^b_{\quad d} R^e_{\quad f} \nabla^2 R^{ef}_{\quad ab},
\]

(7.117)

To show that these are the only tensor structures that contribute to the three point functions, consider the first term which can be shown to be,

\[
R^{ab}_{\quad cd} R^{cd}_{\quad ef} \nabla^2 R^{ef}_{\quad ab} = \nabla_m (\Delta R^{ab}_{\quad cd} \Delta R^{cd}_{\quad ef} \nabla^m \Delta R^{ef}_{\quad ab}) - 2 \nabla_m \Delta R^{ab}_{\quad cd} \nabla^m \Delta R^{ef}_{\quad ab} \Delta R^{cd}_{\quad ef},
\]

(7.118)

where the overall factor of 2 comes because of \( \nabla \) acting on any term other than \( \nabla \Delta R \) are equivalent. Similarly it can be shown for the second term as well.

These terms have additional contribution to the coefficients \( t_2 \) and \( t_4 \) but \( C_T \) remains unaffected.

The coefficients \( c_i \) and \( \tilde{c}_i \) in (7.115) are replaced by their effective values as,

\[
t_2 = \frac{d(d-1)}{c_1 + 2(d-2)c_6'} [2c'_7 - 12(3d+4)c'_8 + 3(7d+4)c''_8],
\]

\[
t_4 = \frac{6d(d^2-1)(d+2)}{c_1 + 2(d-2)c'_6} (2c''_7 - c''_8),
\]

(7.119)

where \( c'_7 = \tilde{c}_7 + 2de_1 \) and \( c''_8 = \tilde{c}_8 + 2de_2 \).

We mention here that although we leave the analysis for the general \( \nabla \ldots \nabla \Delta R \ldots \Delta R \) terms for future work, we feel that this pattern will continue to persist so that the \( \nabla \) terms in the action (7.7) will modify the coefficients appearing in the two and the three point functions and the form of \( C_T, t_2, \) and \( t_4 \) will remain the same as in (7.115) with the coefficients being replaced by similar shifted ones as discussed above.

7.7 Application: \( \eta/s \) for higher derivative theories

As an application of the background field expansion method, we calculate the ratio of the shear viscosity and entropy density [18] for higher derivative theories [16]. This can be done in
arbitrary dimensions but for simplicity, we will illustrate for the $d = 4$ plasma. Following [31] we will use the pole method to calculate the shear viscosity where only the near horizon data is important. Following [31] we write the black hole metric as

$$
 ds^2 = \frac{L^2}{4f(z)} \frac{dz^2}{(1-z)^2} + \frac{r_0^2}{L^2(1-z)} [-f(z)dt^2 + (dx_1 + \phi(t)dx_2)^2 + dx_2^2 + dx_3^2].
$$

(7.120)

To compute the shear viscosity and the entropy density we need to construct the horizon perturbatively by solving the equations of motion for the higher derivative action order by order in coordinate distance from the horizon but exactly in the couplings. The solution can be written as

$$
 f(z) = 2z + f_2 z^2 + f_3 z^3 + \ldots
$$

(7.121)

where $f_2$ and $f_3$ are functions of the coefficients appearing in the action. The factor of 2 fixes the temperature with a particular normalization as

$$
 T = \frac{r_0}{\pi L^2}.
$$

(7.122)

To compute the shear viscosity we have to plug in a perturbation $h_{xy}$ and compute the retarded Green’s function

$$
 G_{R}^{xy,xy}(\omega) = -i \int \theta(t) \langle T^{xy}(t) T^{xy}(0) \rangle e^{-i\omega t},
$$

(7.123)

and finally

$$
 \eta = \lim_{\omega \to 0} \frac{\text{Im} G_{R}^{xy,xy}(\omega)}{\omega}.
$$

(7.124)

We plug in the perturbation corresponding to the shear mode at zero momentum corresponding to the change of basis

$$
 dx_1 \rightarrow dx_1 + \phi(t) dx_2.
$$

(7.125)

Plugging this into the action (7.7), we get

$$
 S_{\phi^2} = \int d^5 x (A_1 \phi_{\omega} \phi'_{-\omega} + A_2 \phi''_{\omega} \phi'''_{-\omega}),
$$

(7.126)

where $A_1$ and $A_2$ are function of the coefficients in the action (7.7). Following [12, 31], we apply the pole method for any general action of the form

$$
 S_{\phi^2} = \int d^4 x dz \mathcal{L}^{(2)}_{\phi} (\partial_x \phi, \partial_t \phi),
$$

(7.127)

using which

$$
 \eta = -8\pi T \lim_{\omega \to 0} \frac{\text{Res}_{z=0} \mathcal{L}^{(2)}_{\phi|z=\omega T}}{\omega^2}.
$$

(7.128)

12See also [32].
Putting in $\phi(t) = e^{-i\omega t}$ we thus extract the coefficient of $1/z$ term and expanding up to quadratic orders in $\omega$, we finally get,

$$\eta = r_0^3(A_1 + B_1 f_2 + C_1 f_2^2 + C_3 f_3),$$

(7.129)

where the coefficients $A_1, B_1, C_1$ and $C_3$ are functions of the coefficients in (7.7). Similarly the entropy density for the higher derivative action is computed using the Wald formula and takes on the form

$$s = 4\pi r_0^3(A_2 + B_2 f_2 + C_2 f_2^2).$$

(7.130)

Note that in the above expressions for both $\eta$ and $s$, we have set the AdS radius $\tilde{L} = 1$. The deviation of the $\eta/s$ ratio from the KSS bound [18] for the action (7.7) corresponding to the case when $t_4 = 0$ and in the absence of $O((\Delta R)^3)$ terms is simply given by

$$(\frac{\eta}{s} - \frac{1}{4\pi})s = -2c_6r_0^3.$$  

(7.131)

The explicit form of $\eta$ and $s$ are given in the appendix (D.5) for a general $R^2$ theory where it is shown that for particular values of the coupling constants of the general $R^2$ theory, the ratio can be driven to zero. As another example we quote the results for the $W^3$ gravity below where the lower bound for $\eta/s$ is much lower than the KSS bound.

Example: $W^3$ theory

In [12], a specific six derivative theory was considered which led to equations of motion for fluctuations which were second order in radial derivatives. The motivation was to consider putting bounds on $\eta/s$ using the positive energy constraints as well as comparing these with the causality constraints. It was found that the positive energy constraints bounded the couplings and led to $(\eta/s)_{min} \approx 0.414/4\pi$. In light of our general analysis, we will consider the following six derivative Lagrangian [10] which also leads to $t_4 \neq 0$ and we will put bounds on the couplings. [10] had already considered this action perturbatively in the couplings:

$$S = \int d^5x \sqrt{g}[R + \frac{12}{\tilde{L}^2} + \frac{1}{2} \lambda W^2 + \tilde{L}^4 \mu W^3],$$

(7.132)

where $W^2 = C_{abcd}C^{abcd}$ and $W^3 = C_{abcdC^{ef}}C^{ef}$ with $C_{abcd}$ being the Weyl tensor. If we expand this action around the AdS background to get (7.7), then the coefficients of (7.7) for this action are given by $c_0 = -8c_1, c_1 = 1, c_4 = \frac{5}{6}, c_5 = -\frac{11}{3}, c_6 = \lambda, c_7 = \frac{4}{3}, c_8 = -4\mu, c_9 = \frac{8\mu}{3}, c_4 = \frac{64\mu}{27}, c_5 = -\frac{14\mu}{9}, c_6 = \frac{7\mu}{3}, c_7 = \mu$ and $c_8 = 0$. Note that for $W^3$ gravity $f_\infty = 1$ and $\tilde{L} = 1$. The coefficients $C_T, t_2$ and $t_4$ take the form

$$C_T = 2(1 + 4\lambda), t_2 = \frac{24(\lambda - 96\mu)}{1 + 4\lambda}, t_4 = \frac{4320\mu}{1 + 4\lambda}.$$  

(7.133)
Using the constraints for $t_2$ and $t_4$ listed in [12], we find that $\lambda$ and $\mu$ are bounded (see figure(7.7)). The shear viscosity and the entropy density for this action takes the form

$$
\eta = \frac{r_0^3}{6}[3 - 6(1 + 2f_2)\lambda - 16(7 - 40f_2 + 16f_2^2 + 36f_3)\mu],
$$

$$
s = \frac{2\pi r_0^3}{3}[3 + 6(1 - 2f_2)\lambda + 16(1 - 2f_2)^2 \mu],
$$

where $f_3$ is given in terms of $f_2$ by,

$$
f_3 = \frac{270 - 64\mu + 18\lambda + f_2(216 - 171\lambda + 656\mu) - 6f_2^2(9 - 42\lambda + 304\mu) + 4f_2^3(9\lambda + 368\mu) + 128f_2^2\mu}{36(-9 - 6(1 - 2f_2)\lambda + 16(1 - 2f_2)^2 \mu)},
$$

where $f_2$ satisfies

$$
64(1 - 2f_2)^3\mu + 36(6 + f_2) - \lambda + 4(1 - f_2)f_2\lambda = 0.
$$

This equation has three roots and we will choose the correct root as the one which for the Einstein case goes to $f_2 = -1$. Substituting for the Einstein value of $f_2$ we also get that $f_3 = 0$ in the Einstein limit. We present the bounds on $\lambda$ and $\mu$ in Fig.(7.7) obtained from the causality

---

**Figure 7.1:** $\lambda$ vs $\mu$ plot. The horizontal line corresponds to $\mu = 0$. $(\eta/s)_{\text{min}} = 0.55/4\pi$ for $\mu = 0$ and $(\eta/s)_{\text{min}} = 0.17/4\pi$ for $\mu \neq 0$. 
The minimum values of $\eta/s$ for $\mu \neq 0$ lie close to the uppermost vertex of the triangle. For Weyl squared gravity $\mu = 0$ and constraints give $-1/12 < \lambda < 1/4$. This is presented as a single line interval in the $\lambda - \mu$ plot. The minimum value of $\eta/s$ corresponds to $\lambda = 1/4$ and $\mu = 0$ which is at the extreme right end of the interval. The minimum value of $\eta/s$ for the $W^3$ gravity is given by

$$\eta/s \approx \frac{0.17}{4\pi},$$

for $\lambda = 1/2$, and $\mu = 1/192$ which is the uppermost vertex of the triangle. For $\mu = 0$, i.e., for Weyl squared gravity the minimum value of the ratio is $\eta/s \approx 0.55/4\pi$.

Thus even though $\langle \epsilon \rangle > 0$ for general $R^2$ theory, the $\eta/s$ ratio can be driven to zero as we show in appendix (D.5). Further for $W^2$ theory we can see that the bound goes down to about 55% of the KSS value whereas for $W^2 + W^3$ theory it is 17% of the KSS bound. There are non-unitary modes in this theory. So it appears that unitarity is not a prerequisite for a bound. As in [12], there could be potential plasma instabilities and it may be interesting to analyze these.

### 7.8 Discussion

In this chapter we have computed one, two and three point functions for a general gravity Lagrangian of the form $L(g^{ab}, R_{cdef}, \nabla_a R_{bcde})$. We explained that the coefficient appearing as the proportionality between the renormalized stress tensor and the bulk metric is related to $C_T$, the coefficient appearing in the two point function of stress tensors. Further we saw how this relates to B-type anomaly coefficients in even dimensions. We also computed three point functions for bulk Lagrangians of the above form in arbitrary dimensions.

Our general form of the action given in eq. (7.7) packages the A-type anomaly coefficient (or its analog in odd dimensions) into $c_1$ while $C_T$ is given in terms of $c_1, c_6$. Again we emphasize that all these coefficients themselves depend on all higher derivative terms that appear in the original bulk Lagrangian. This simple separation of the A-type anomaly coefficient as a proportionality constant in front of $\Delta R$ makes it very tempting to think that this is a useful starting point for a general proof of holographic version of the $a$-theorem [33] in arbitrary dimensions. We can speculate how this may work: First note that the background around which we are expanding could be either the AdS in the ultraviolet or the AdS in the infrared. This means that the respective background expanded Lagrangians must be equal to one another. If there was a
matter sector as well, it makes sense to do a background expansion of this sector where we will use the background for the matter fields to be their values in AdS, for example for a scalar field this will be a constant (different constants in the UV and IR). We thus have a natural separation between the gravity sector and the matter sector—this was one of the vexing issues in the current literature on holographic c-theorems [34]; namely how does one define any energy condition if matter couples to the higher curvature terms. Thus we can envisage a situation where on the LHS we have a term proportional to \((a_{UV} - a_{IR}) R\) plus other curvature terms while on the RHS we can place the difference between the UV and IR matter Lagrangians. It is very tempting to speculate that \((a_{UV} - a_{IR}) > 0\) is necessary for there to be no non-unitary modes on the LHS arising from expanding \(R\) which in turn is necessary (but may not be sufficient) so that there are no non-unitary modes in the matter Lagrangian. It will be nice to work this out in complete detail as this will shed light on how the proof of the \(a\)-theorem may work in arbitrary dimensions.

Another important question is to extend our methods and results to four point stress tensor correlation functions. As we pointed out in the introduction, while the A-type trace anomaly in 4d is related to two point and three point functions, in higher dimensions it appears to depend on higher point correlation functions. Also in 3d since there is no analog of \(t_2\), it is unclear if the analog of the A-type trace anomaly (proportional to \(c_1\)) can be extracted from local correlation functions at all—this appears to be consistent with recent claims in [35]. The general forms for \(t_2\) and \(t_4\) that we have derived also seem to suggest that in order to relate the A-type anomaly coefficient in dimensions higher than 4 to the coefficients appearing in correlation functions will need at least four point functions. Furthermore, it could well be that the coupling constants for higher derivative theories are further constrained by considering four point functions\(^{13}\). These reasons are sufficient motivation to look at the four point functions in the general gravity theories we have considered in this chapter. May be the techniques developed in [36] could help us out here.

It will be interesting to extend our results to completely general bulk Lagrangians of the form \(\mathcal{L}(g^{ab}, R_{cdef}, \nabla_a R_{bcde}, \nabla_a (\nabla_b) R_{cdef}, \cdots)\). We expect that for the one, two and three point functions, the simple features we have found will continue to hold. Finally, it should be possible to extend our methods to study correlation functions which involve the massive graviton modes and \(T_{\mu\nu}\) [37].

\(^{13}\)It will also be interesting to compare how constraints from entanglement entropy [38] compare with these ones.
Bibliography


Chapter 7. Holographic stress tensor at finite coupling


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Chapter 8

Concluding remarks

This thesis is aimed at a better understanding of constraints on the conformal field theory spectrum both from the perspective of the Conformal Bootstrap Program and holographic principles. Below we will give the open questions related to each of these works and future directions which can be addressed.

We started with some basic introductions to the Conformal field theory in chapter 2, where we discussed the ideas leading up to the conformal bootstrap program. In the next chapter 3, we provided some details on the holographic duality which relates gravity in AdS bulk space time to a CFT living on the boundary. Various aspects of the duality are discussed in this chapter with the canonical example in mind.

In chapter 4, we discussed the first problem of constraining the large spin sector of the spectrum that dominates the light cone regime of the bootstrap equation. For this we considered only identical external scalars, and found that with a stress tensor in the spectrum, the anomalous dimensions of the large spin sector can be written as an exact polynomial function of the twist when the spin is much larger than the twist of the operators. From the holographic side, this corresponds to the calculation of the Shapiro time delay in the supergravity limit. Albeit we show a calculation of the other regime in Appendix A, when the twist is much larger than the spin of the operators, we believe that the results are not complete, since we should be taking into account the other modes, as the massive scalar modes (neglected in the SUGRA limit, coming from the KK compactification) and the string modes that dominate this regime. It will be interesting to see what happens when these modes are also taken into account. Certainly these additional modes will deform the theory away from the SUGRA and lead to the question of whether the correct theory for this limit is stringy or not. Hence to emphasize, the information about this regime will provide an answer to the question about whether string theory is a potential candidate towards the UV completion of the low energy theory and if so, then which type of string theory.

Chapter 5 discusses an extension of the chapter 4, where the results where extended to a CFT of general dimension. To extract an exact behavior of the anomalous dimension for the large
spin operators, as a function of the twist for general dimensions appear difficult. Nevertheless, we were able to extract the leading universal behavior of the anomalous dimension when the twist is much larger than the dimensions of the external scalars but still much smaller than the spin. This behavior, as we pointed out, matches with the existing holographic calculation of the Shapiro time delay in general $d + 1$ dimensional bulk. Here also, the questions pertaining to the other limit applies as discussed in the previous paragraph. Apart from that, we were also able to partly answer the question about the universality aspects of certain results pertaining to a general $d$ dimensional CFT.

While the above chapters describe particular type of CFTs at strong coupling (and large $N$ expansion for holographic CFTs), a different story dominates field theories in the other regime with weak coupling and finite dimensions of the internal symmetry group. Here the conventional QFT approaches as the $\epsilon-$expansion (tied to the diagrammatic perturbation theory) proves to be of much needed help although the calculations become tedious as one goes up in the order of the $\epsilon$ mainly due to difficulties with loop diagrams and multiple structures. As we pointed out in chapter 6, using the seminal approach of Polyakov, one can obtain a certain set of algebraic constraint relations by equating the unitary crossing symmetric four point correlator with the consistency condition arising from the OPE expansion. These constraint equations are much lucid than the functional constraint equations obtained in the conventional bootstrap approach and can be solved order by order in $\epsilon$. This seemingly different approach is not that different since instead of imposing the crossing symmetry after assuming completeness condition as done in modern conventional bootstrap approach, we are first demanding crossing symmetry and then imposing the completeness condition. So this is the same approach in a reverse engineered style. Clearly this approach is more robust than the modern approach although still in its infancy. A more nicer way to perform these calculations is via Mellin space where one can build up the four point correlator, an approach which is the main focus of the second part of Polyakov’s work. In chapter 6, as a first extension of this novel piece of work, we extended this approach à la Polyakov, to correctly match the anomalous dimensions of a certain subset of scalar exchange operators up to one higher order in $\epsilon$ with those existing in literature. Since this approach does not entail taking the limit of large rank of the internal symmetry group, this has potential application to theories with finite dimension of the gauge group and with intermediate couplings.

A further connection between the chapters, 4, 5 and 6 is the seemingly two different ways of bootstrap approach, used to handle the different regimes (strong and weak coupling) of CFT. This points to the fact that bootstrap can be used to put contraints on the spectrum in general. It will be actually nice to interpolate between these two regimes using the bootstrap to constrain theories with both finite coupling and finite rank of the gauge (internal symmetry) group.

In comparison to the other chapters, the last chapter might seem a little off beat. But the motivation in the introduction of chapter 7 makes this clear that the work presented in the chapter is a part of the bigger goal of finding the four point functions (of stress tensors) in
CFTs with finite coupling. The finite coupling in the boundary theory side, corresponds to generic higher derivative theories of gravity in the bulk dual and is our starting point of the chapter. We have not computed the four point functions of gravitons in the bulk, but as a first step towards this, we have analyzed the possible structure of the three point functions of the stress tensor in the bulk, which corresponds to the graviton three point vertex, albeit in a special bulk environment (shock-wave background). These three point functions take into account, the effects of the finite 't Hooft coupling through specific dependencies on the bulk higher derivative couplings. Thus in this chapter we computed the one, two and three point functions of the stress tensor in the bulk for generic higher derivative theories of gravity. In general we showed how these correlation functions depended on the higher derivative couplings and how the universal behavior of these coefficients are related to certain types of anomalies in the boundary theory. These three point functions can be used as building blocks for constructing four point functions (in a finitely coupled boundary theory) involving stress tensors. Moreover, various other mixed four point correlators involving stress tensors can also be constructed and we want to be ready with the bootstrap formalism, for these mixed correlators.

In conclusion, while we emphasized that the bootstrap approach puts constraints on any CFT in general (both the strongly and weakly coupled domain), we believe that during the course of the interpolation between these two theories, the other modes (like massive KK modes and other string modes) will show up in the picture. Moreover if one believes that the AdS/CFT duality is fairly generic, then bootstrap might be the only tool to demonstrate this duality rigorously.

Finally, continuing in this path might lead to the uniqueness of string theory. Hence to conclude this thesis, there are a number of interesting questions waiting to be explored!
Appendix A

Details of Chapter 4

A.1 Calculation details

To clearly see the expressions for the anomalous dimensions discussed in the main text we now take a mathematical detour a little to explain some of the steps and the useful formulas that goes into the derivation of the above expressions. Note that in the following calculations we will not put the overall factor of $4P_m$ for convenience. Each of the above expressions use the summation of the generic type

$$a(x,m,\epsilon) = \sum_{k=0}^{m} \frac{\Gamma(x+k)}{k!\Gamma(x)} \epsilon^k.$$  \hfill (A.1)

Using the integral representation of the $\Gamma$-function, the summation on the rhs can be converted into,

$$a(x,m,\epsilon) = \frac{1}{\Gamma(x)} \int_{0}^{\infty} dt \ e^{-t} \sum_{k=0}^{m} \frac{t^{x+k-1}}{k!} \epsilon^k.$$ \hfill (A.2)

The summation inside the integral can be written as,

$$\sum_{k=0}^{m} \frac{t^{x+k-1}}{k!} \epsilon^k = e^{t} t^{x-1} \frac{\Gamma(m+1, \epsilon t)}{\Gamma(m+1)} = e^{t} t^{x-1} \int_{\epsilon t}^{\infty} z^{m} e^{-z} dz,$$ \hfill (A.3)

where $\Gamma(a,x)$ is the incomplete Gamma function given by $\Gamma(a,x) = \int_{x}^{\infty} z^{a-1} e^{-z} dz$. Thus the function $a(x,m)$ becomes after the above substitution as,

$$a(x,m,\epsilon) = \frac{1}{\Gamma(x)\Gamma(m+1)} \int_{0}^{\infty} dt \ e^{(\epsilon-1)t} t^{x-1} \int_{\epsilon t}^{\infty} dz \ z^{m} e^{-z}.$$ \hfill (A.4)

At this point we do a change of variable from $z$ to $z = y + \epsilon t$ whereby we notice that the limits of the integral on $z$ changes to $y = 0$ and $y = \infty$ respectively. Thus we get,

$$a(x,m,\epsilon) = \frac{1}{\Gamma(x)\Gamma(m+1)} \int_{0}^{\infty} dt \ dy \ (y+\epsilon t)^{m} e^{-(t+y)t^{x-1}}.$$ \hfill (A.5)
Whatever summation formulas we have derived in the text are linear combinations of the above function and its derivatives. For example,

$$a(x, m, \epsilon = 1) = \frac{\Gamma(x)\Gamma(m + x + 1)}{\Gamma(x + 1)\Gamma(m + 1)}.$$  \hspace{1cm} (A.6)

Again a polynomial arranged like,

$$m \sum_{k=0}^{m} \left[ c_0 + c_1 k + c_2 k(k - 1) + c_3 k(k - 1)(k - 2) + c_4 k(k - 1)(k - 2)(k - 3) + \cdots \right] \frac{\Gamma(k + x)}{k!\Gamma(x)}$$

$$= c_0 a(x, m, \epsilon)\big|_{\epsilon=1} + c_1 \partial_\epsilon a(x, m, \epsilon)\big|_{\epsilon=1} + c_2 \partial^2_\epsilon a(x, m, \epsilon)\big|_{\epsilon=1} + c_3 \partial^3_\epsilon a(x, m, \epsilon)\big|_{\epsilon=1} + c_4 \partial^4_\epsilon a(x, m, \epsilon)\big|_{\epsilon=1} + \cdots,$$  \hspace{1cm} (A.7)

where,

$$\partial^i_\epsilon a(x, m, \epsilon)\big|_{\epsilon=1} = \sum_{k=0}^{m} k(k - 1) \cdots (k - i + 1) \frac{k!}{(x + k)!} = \frac{\Gamma(m + x + 1)}{(x + m + \tau_m + 2\epsilon - 2)!\Gamma(x)}.$$  \hspace{1cm} (A.8)

### A.2 Verification of some useful formulae

With the definitions of the formula in the previous section we can now apply them to our cases specific to the exchange of the twist-2 scalar and a spin-2, twist-2 field.

#### A.2.1 $\ell_m = 0$ and $\tau_m = 2$

We will first deal with the case of a twist-2 scalar exchange. The formulas are much simpler for this case.

1. $$\frac{(-m)^2}{k!} \frac{k!}{k!(1 - \ell_m - m - \frac{\tau_m}{2})} \frac{\Gamma(-2 + k + \Delta)}{k!\Gamma(-2 + \Delta)}.$$  \hspace{1cm} (A.9)

This formula needs no verification. We can simply put $\ell_m = 0$ and $\tau_m = 2$ to see that the \textit{rhs} is produced.

2. $$\sum_{k=0}^{m} \frac{\Gamma(x + k)}{k!\Gamma(x)} = \frac{\Gamma(1 + m + x)}{\Gamma(1 + m)\Gamma(1 + x)}.$$  \hspace{1cm} (A.10)

To see this we recall from the previous section that

$$\sum_{k=0}^{m} \frac{\Gamma(x + k)}{k!\Gamma(x)} = a(x, m, \epsilon = 1).$$  \hspace{1cm} (A.11)

Performing the integrals at $\epsilon = 1$, fixes the form on the \textit{rhs} of the above formula.
In this case the coefficients \(a_{n,m}\) are given by,

\[
a_{n,m} = -\left(-1\right)^{m+n} \frac{(\Delta_\phi - 1)\Gamma(n + 1)\Gamma(2\Delta_\phi + n + m - 3)}{8m!(n-m)!\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)}.
\]  

We will now use the reflection formula for the \(\Gamma\)-functions to obtain,

\[
\Gamma(m + \Delta_\phi - 1) = (-1)^{-(m+1)} \frac{\pi}{\sin(\pi\Delta_\phi)\Gamma(2 - \Delta_\phi - m)}.  
\]  

Separating out the \(m\) independent parts and using the integral representation of the product of the \(\Gamma\)-functions given by,

\[
\Gamma(n + m + 2\Delta_\phi - 3)\Gamma(2 - \Delta_\phi - m) = \int_0^\infty \int_0^\infty dx dy e^{-(x+y)} x^{n+m+2\Delta_\phi-4} y^{-m+1-\Delta_\phi},
\]
we can perform the sum over \(m\) to get,

\[
\sum_{m=0}^n (x/y)^m \frac{n!}{m!(n-m)!} = \frac{1}{n!} \left(\frac{x + y}{y}\right)^n \equiv b(n, x, y).
\]

Hence the coefficient \(\gamma_n\) associated with the anomalous dimensions become,

\[
\gamma_n = \frac{(-1)^{n+1} \sin(\pi\Delta_\phi) (\Delta_\phi - 1)\Gamma(n + 1)\Gamma(\Delta_\phi)}{8\Gamma(n + 2\Delta_\phi - 3)} \int_0^\infty dx dy b(n, x, y)e^{-(x+y)} x^{n+2\Delta_\phi-4} y^{1-\Delta_\phi}.
\]

Using the transformation of variables for \(x = r^2 \cos^2 \theta\) and \(y = r^2 \sin^2 \theta\) and performing the integral over only the first quadrant, the integration limits change from \(r = 0\) to \(r = \infty\) and \(\theta = 0\) to \(\theta = \pi/2\). The integral thus becomes,

\[
\int_0^\infty dx dy b(n, x, y)e^{-(x+y)} x^{n+2\Delta_\phi-4} y^{1-\Delta_\phi} = -\frac{(-1)^{n-1} \pi \csc(\pi\Delta_\phi)\Gamma(n + 2\Delta_\phi - 3)}{\Gamma(n + 1)\Gamma(\Delta_\phi - 1)}.
\]

Putting this with the overall factors we get,

\[
\gamma_n = -\frac{1}{8}(\Delta_\phi - 1)^2.
\]

which is independent of \(n\). Here we have not taken into account the overall factor of \(4P_m\) that we should multiply with the expression for \(\gamma_n\) to match the result with the main text.

\section*{A.2.2 \(\ell_m = 2\) and \(\tau_m = 2\)}

We list below the derivation of important formulas required pertaining to this case.
Appendix A. Details of Chapter 4

1. \[
\frac{(-m)^2(-1-\ell_m+\Delta-\frac{zm}{2})_k}{(1-\ell_m-m-\frac{zm}{2})^2_k k!} = \frac{(1-k+m)^2(2-k+m)^2\Gamma(-4+k+\Delta)}{(1+m)^2(2+m)^2\Gamma(1+k)\Gamma(-4+\Delta)}. \tag{A.20}
\]

As in the scalar case we put \(\tau_m = 2\) and \(\ell_m = 2\) for this case to retrieve the rhs of the above formula.

2. \[
\sum_{k=0}^{m} \frac{(1-k+m)^2(2-k+m)^2 \Gamma(x+k)}{k!(1+m)^2(2+m)^2} \frac{\Gamma(x)}{\Gamma(x)} = \frac{4(6m^2 + 6m(3+x) + (3+x)(4+x))\Gamma(3+m+x)}{(1+m)(2+m)\Gamma(3+m)\Gamma(5+x)} . \tag{A.21}
\]

To get to this, we will appeal to (A.7), by noticing that the factor \((1-k+m)^2(2-k+m)^2\) can be arranged as,

\[
(1-k+m)^2(2-k+m)^2 = Ak(k-1)(k-2) + Bk(k-1) + Ck + Dk + E, \tag{A.22}
\]

where \(A = 1\), \(B = -4m\), \(C = 6m^2 + 6m + 2\), \(D = -4(m+1)^3\) and \(E = (2+3m+m^2)^2\).

Thus the sum becomes,

\[
\sum_{k=0}^{\infty} \frac{(1-k+m)^2(2-k+m)^2 \Gamma(x+k)}{(m+1)^2(m+2)^2} \frac{\Gamma(x)}{k!\Gamma(x)} = A\partial_{\epsilon}^2 a(x,m,\epsilon)|_{\epsilon=1} + B\partial_{\epsilon}^3 a(x,m,\epsilon)|_{\epsilon=1} + C\partial_{\epsilon}^2 a(x,m,\epsilon)|_{\epsilon=1} + D\partial_{\epsilon} a(x,m,\epsilon)|_{\epsilon=1} + E a(x,m,\epsilon)|_{\epsilon=1}. \tag{A.23}
\]

We know how the each of the terms go by looking at (A.8). By combining the coefficients we find that the rhs is produced.

3. \[
\gamma_n = \sum_{m=0}^{n} a_{n,m}. \tag{A.24}
\]

We will now prove the final piece of the analytic puzzle as follows. First note that \(a_{n,m}\) for \(\ell_m = 2\) and \(\tau_m = 2\) is given in a closed form expression as

\[
a_{n,m} = (-1)^{n+m} \frac{15(6m^2 + 6m(\Delta_\phi - 1) + \Delta_\phi(\Delta_\phi - 1))}{4\Delta_\phi} \times \frac{\Gamma(n+1)\Gamma(n+m+2\Delta_\phi-3)}{m!(n-m)\Gamma(m+\Delta_\phi-1)\Gamma(n+2\Delta_\phi-3)}. \tag{A.25}
\]

We will now use the reflection formula for the \(\Gamma\)-functions to obtain,

\[
\Gamma(m + \Delta_\phi - 1) = (-1)^{-(m+1)} \frac{\pi}{\sin(\pi\Delta_\phi)\Gamma(2 - \Delta_\phi - m)}. \tag{A.26}
\]
Appendix A. Details of Chapter 4

Separating out the \(m\)-independent parts we have

\[
\gamma_n = \frac{(-1)^{n+1} \sin(\pi \Delta_\phi)}{\pi} \frac{15\Gamma(n+1)\Gamma(\Delta_\phi)}{\Gamma(n+2\Delta_\phi-3)4\Delta_\phi} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} [6m^2 + 6m(\Delta_\phi - 1) + \Delta_\phi(\Delta_\phi - 1)] \Gamma(n + m + 2\Delta_\phi - 3) \Gamma(2 - \Delta_\phi - m). \tag{A.27}
\]

The integral representation of the product of the two \(\Gamma\)-functions is given by

\[
\Gamma(n + m + 2\Delta_\phi - 3) \Gamma(2 - \Delta_\phi - m) = \int_{0}^{\infty} \int_{0}^{\infty} dx dy e^{-(x+y)} x^{n+m+2\Delta_\phi-4} y^{-m+1-\Delta_\phi}. \tag{A.28}
\]

Performing the sum over \(m\) inside the integral for a polynomial multiplying the \(\Gamma\)-functions of the form \(f(m) = c_0 + c_1 m + c_2 m^2\) we get,

\[
\sum_{m=0}^{n} \left(\frac{x}{y}\right)^m \frac{f(m)}{m!(n-m)!} = \left(\frac{x+y}{y}\right)^n c_0 (x+y)^2 + c_1 nx(x+y) + c_2 nx(nx+y) \equiv b(n, x, y). \tag{A.29}
\]

Thus the expression for \(\gamma_n\) becomes,

\[
\gamma_n = \frac{(-1)^{n+1} \sin(\pi \Delta_\phi)}{\pi} \frac{15\Gamma(n+1)\Gamma(\Delta_\phi)}{\Gamma(n+2\Delta_\phi-3)4\Delta_\phi} \int_{0}^{\infty} \int_{0}^{\infty} dx dy \ b(n, x, y) e^{-(x+y)} x^{n+2\Delta_\phi-4} y^{1-\Delta_\phi}. \tag{A.30}
\]

Using the transformation of variables for \(x = r^2 \cos^2 \theta\) and \(y = r^2 \sin^2 \theta\) and performing the integral over only the first quadrant, the integration limits change from \(r = 0\) to \(r = \infty\) and \(\theta = 0\) to \(\theta = \pi/2\). Thus, putting the values of \(c_0 = \Delta_\phi(\Delta_\phi - 1), c_1 = 6(\Delta_\phi - 1)\) and \(c_2 = 6\), we have

\[
\int_{0}^{\infty} \int_{0}^{\infty} dx dy \ b(n, x, y) e^{-(x+y)} x^{n+2\Delta_\phi-4} y^{1-\Delta_\phi} \equiv - \frac{(-1)^{n-1} \pi \csc(\pi \Delta_\phi)\Gamma(n+2\Delta_\phi-3)}{\Gamma(n+1)\Gamma(\Delta_\phi+1)} [6n(n+2\Delta_\phi - 3)(2 - \Delta_\phi + n(n+2\Delta_\phi - 3)) + \Delta_\phi(\Delta_\phi - 1)(\Delta_\phi(\Delta_\phi - 1) + 6n(n+2\Delta_\phi - 3))]. \tag{A.31}
\]

Multiplying this by the overall \(n\)-dependent factors outside we have,

\[
\gamma_n = - \frac{15}{4\Delta_\phi^2} [6n^4 + \Delta_\phi^2(\Delta_\phi - 1)^2 + 12n^3(2\Delta_\phi - 3) + 6n^2(11 - 14\Delta_\phi + 5\Delta_\phi^2) + 6n(2\Delta_\phi - 3)(\Delta_\phi^2 - 2\Delta_\phi + 2)], \tag{A.32}
\]

which is the precise formula for \(\gamma_n\) in \(d = 4\) dimensions. Note that the final expression for \(\gamma_n\) derived above needs to be multiplied by an overall factor of \(4P_m\) to match with that in the main text.
Appendix A. Details of Chapter 4

A.3 \( n \) dependence of \( \gamma_n \) for \( \ell_m > 2 \)

In this section we will give an overview on the leading \( n \) dependence of the coefficients of the anomalous dimensions \( \text{viz.} \ \gamma_n \). We will consider two cases with twist-2 and spins \( \ell_m = 4, 6 \). For \( \ell_m = 4 \), the coefficients \( a_{n,m} \) are given by,

\[
a_{n,m} = -\frac{315P_m(-1)^{m+n}\Gamma(n+1)\Gamma(\Delta_\phi)^2\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(\Delta_\phi + 3)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)} \\
x [70m^4 + 140m^3(\Delta_\phi - 1) + 10m^2(9\Delta_\phi^2 - 15\Delta_\phi + 11) + 10m(2\Delta_\phi^3 - 3\Delta_\phi^2 + 5\Delta_\phi - 4) \\
\Delta_\phi(\Delta_\phi^2 - 1)(\Delta_\phi + 2)\].
\]  
(A.33)

To calculate the leading \( n \) dependence in the coefficient \( \gamma_n \), we take the leading term proportional to \( m^4 \) in \( a_{n,m} \) and do the sum over \( m \) to get,

\[
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{22050P_m n^8}{\Delta_\phi^2(\Delta_\phi + 1)^2(\Delta_\phi + 2)^2} - \cdots .
\]  
(A.34)

Thus the leading \( n \) dependence of the coefficients \( \gamma_n \) for \( \ell_m = 4 \) is \( \sim -n^8 \). Similarly for \( \ell_m = 6 \), the coefficients \( a_{n,m} \) are given by,

\[
a_{n,m} = -\frac{6006P_m(-1)^{m+n}\Gamma(n+1)\Gamma(\Delta_\phi)^2\Gamma(2\Delta_\phi + m + n - 3)}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(\Delta_\phi + 5)\Gamma(\Delta_\phi + m - 1)\Gamma(2\Delta_\phi + n - 3)} \\
x [924m^6 + 2772m^5(\Delta_\phi - 1) + 210m^4(15\Delta_\phi^2 - 27\Delta_\phi + 26) + 420m^3(\Delta_\phi - 1) \\
(4\Delta_\phi^2 - 5\Delta_\phi + 15) + 42m^2(10\Delta_\phi^4 - 20\Delta_\phi^3 + 95\Delta_\phi^2 - 145\Delta_\phi + 88) + 42m(\Delta_\phi^5 \\
15\Delta_\phi^3 - 30\Delta_\phi^2 + 38\Delta_\phi - 24) + (\Delta_\phi + 4)(\Delta_\phi + 3)(\Delta_\phi + 2)(\Delta_\phi + 1)\Delta_\phi(\Delta_\phi - 1)\].
\]  
(A.35)

Again, we take the leading term in \( m \) in \( a_{n,m} \) and sum over \( m \) to get,

\[
\gamma_n = \sum_{m=0}^{n} a_{n,m} = -\frac{5549544P_m n^{12}}{\Delta_\phi^2(\Delta_\phi + 1)^2(\Delta_\phi + 2)^2(\Delta_\phi + 3)^2(\Delta_\phi + 4)^2} - \cdots .
\]  
(A.36)

All the above expressions for \( \gamma_n \) are up to overall normalization constants. Thus for a generic \( \ell_m \) we find that the coefficient \( \gamma_n \) has an \( n \) dependence given by,

\[
\gamma_n \sim -n^{2\ell_m} .
\]  
(A.37)

A.4 Sub leading correction at large \( \ell \) and large \( n \)

In this section we will provide an argument why it is sufficient to consider the expansion of the Bessel functions in (4.49) up to the order we did. To see that, consider the differential equation
for the hypergeometric function \( _2F_1(\beta/2, \beta/2; \beta; 1 - z) \),

\[
z(1-z)\frac{d^2w}{dz^2} + \left[1 - (\beta + 1)z\right]\frac{dw}{dz} - \frac{\beta^2}{4}w = 0. \quad (A.38)
\]

Here \( \beta = \tau + 2\ell \). The large \( \ell \) limit is same as the large \( \beta \) limit. We can then expand the solution in the form,

\[
w = w_0 + \frac{1}{\beta}w_1 + O \left( \frac{1}{\beta^2} \right). \quad (A.39)
\]

Consider the change of variables as \( y = \beta^2 z \) in which the differential equation takes the form,

\[
y \left(1 - \frac{y}{\beta^2}\right)\frac{d^2w}{dy^2} + \left[1 - (\beta + 1)\frac{y}{\beta^2}\right]\frac{dw}{dy} - \frac{1}{4}w = 0. \quad (A.40)
\]

The differential equations for the functions \( w_0 \) and \( w_1 \) are given by,

\[
y w_0'' + w_0' - w_0 = 0, \quad y w_1'' + w_1' - w_1 - 2yw_0' = 0. \quad (A.41)
\]

The solutions are given by,

\[
w_0 = c_0 K_0(2\sqrt{y}), \quad w_1 = f_1(2\sqrt{y}), \quad (A.42)
\]

where \( \sqrt{y} = \beta \sqrt{z} \). Thus for large \( \ell \) we can expand the full solution \( w(y) \) as,

\[
w(y) = c_0 K_0(2\sqrt{y}) + \left(\frac{1}{2\ell} - \frac{n}{2\ell^2}\right)f_1(2\sqrt{y}) + O \left( \frac{1}{\ell^3} \right). \quad (A.43)
\]

Further now if we consider the expansion of the variable \( y \),

\[
w(z) = c_0 K_0[(2\ell + \tau)\sqrt{z}] + \left(\frac{1}{2\ell} - \frac{n}{2\ell^2}\right)f_1[(2\ell + \tau)\sqrt{z}] + O \left( \frac{1}{\ell^3} \right). \quad (A.44)
\]

In the limit \( \ell \gg n \gg 1 \), we have,

\[
w(z) = c_0 (K_0(2\ell\sqrt{z}) - \sqrt{z\tau}K_1(2\ell\sqrt{z})) + O(n/\ell^2). \quad (A.45)
\]

The terms coming from \( w_1 \) are hence sub leading compared to the leading order result in the limit of large \( \ell \). The overall constant \( c_0 \) is given by,

\[
c_0 = \frac{\Gamma(2\ell + \tau)}{\Gamma(\ell + \frac{\tau}{2})^2} = \frac{2^{2\ell+\tau-1}}{\sqrt{\pi}}\ell^{1/2} \left(1 + \frac{2\tau - 1}{8\ell}\right). \quad (A.46)
\]

Combined with the leading order expansion for \( w(z) \) gives (4.49).
A.5 Correction to OPE coefficients for $\ell \gg n \gg 1$

We now turn to the question about what happens to the leading corrections to the OPE coefficients for the $\ell \gg n \gg 1$ case. The starting point of the calculation is,

$$
\sum_{n,\ell} P^{MFT}_{2\Delta_\phi+2n,\ell}\left(\delta P_{2\Delta_\phi+2n,\ell} + \frac{1}{2} \gamma(n, \ell) \frac{\partial}{\partial n}\right) v^n 4^{\ell/2} K_0(2\ell \sqrt{z}) F^{(4)}[2\Delta_\phi + 2n, v] = \sum_{\alpha} A_\alpha v^\alpha, \tag{A.47}
$$

where we are now only considering the terms without the log $v$ term in (4.14). As before we can perform the integration over the spins to eliminate one of the sums. To get the same leading order in $z$ as explained in [7], the coefficients $\delta P_{2\Delta_\phi+2n,\ell}$ should go like,

$$
\delta P_{2\Delta_\phi+2n,\ell} = \frac{c_n}{\ell^{m_n}}. \tag{A.48}
$$

Thus the above equation becomes, after performing the $\ell$ integration,

$$
\frac{1}{8} \Gamma\left(\Delta_\phi - \frac{\tau_m}{2}\right) 2 \sum_{n} q_{\Delta_\phi,n} \left[C_n + \frac{1}{2} \gamma_n \frac{\partial}{\partial n}\right] v^n F^{(4)}[2\Delta_\phi + 2n, v] = \sum_{\alpha} A_\alpha v^\alpha. \tag{A.49}
$$

Acting the derivatives of $n$ on $v^n$ obtains a $v^n \log v$ term and the terms containing only $v^n$ come from considering,

$$
\frac{1}{8} \Gamma\left(\Delta_\phi - \frac{\tau_m}{2}\right) 2 \sum_{n} q_{\Delta_\phi,n} \left[C_n F^{(4)}[2\Delta_\phi + 2n, v] + \frac{1}{2} \gamma_n \partial_n F^{(4)}[2\Delta_\phi + 2n, v]\right] v^n = \sum_{\alpha} A_\alpha v^\alpha. \tag{A.50}
$$

At this point note that the function $F^{(4)}[2\Delta_\phi + 2n, v] = 2^n F_1(\Delta_\phi+n-1,\Delta_\phi+n-1,\Delta_\phi+2n-2; v)$ has a separate $n$ dependent part coming from the $2^n$. So the $n$-derivative should act on this part as well. Thus equation (A.50) becomes,

$$
\frac{1}{8} \Gamma\left(\Delta_\phi - \frac{\tau_m}{2}\right) 2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q_{\Delta_\phi,n} d_{n,k} (C_n + \gamma_n (\log 2 + g_{n,k})) v^{n+k} = \sum_{\alpha=0}^{\infty} A_\alpha v^\alpha. \tag{A.51}
$$

where the function $g_{n,k}, d_{n,k}$ are defined as,

$$
g_{n,k} = \psi(2\Delta_\phi + 2n - 2) + \psi(n + \Delta_\phi + k - 1) - \psi(\Delta_\phi + n - 1) - \psi(2\Delta_\phi + 2n + k) - (2), \tag{A.52}
d_{n,k} = \frac{(\Delta_\phi + n - 1)_k}{(2\Delta_\phi + 2n - 2)_k k!}, \tag{A.53}
$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. To regroup the terms in (A.51) increasing powers of $v^\alpha$, we set $n+k = \alpha$ and the lhs of the above equation becomes $\sum_{\alpha=0}^{\infty} f_{\alpha,\Delta_\phi} v^\alpha$ where,

$$
f_{\alpha,\Delta_\phi} = \sum_{k=0}^{\alpha} q_{\alpha-k,\Delta_\phi} d_{\alpha-k,k} C_{\alpha-k} + b_{\alpha}, \quad \text{where} \quad b_{\alpha} = \sum_{k=0}^{\alpha} q_{\alpha-k,\Delta_\phi} d_{\alpha-k,k} \gamma_{\alpha-k} (\log 2 + g_{\alpha-k,k}). \tag{A.54}
$$
By equating the two sides of the above equation via \( f_{\alpha, \Delta \phi} = A_\alpha \), we can get the coefficients \( C_n \) once we know the anomalous dimensions \( \gamma_n \). On the \( \text{lhs} \) of (A.50), the coefficients \( A_\alpha \) are determined as follows. We have absorbed the term \( (1 - v)^{\Delta \phi - 1} \) in to the \( \text{lhs} \) of (4.13) to obtain,

\[
(1 - v)^{\tau_m/2 + \ell_m + 1 - \Delta \phi} P_m \Gamma(\ell_m + 2\tau_m/2) \sum_{n=0}^{\infty} \left( \frac{(\ell_m + \tau_m/2)n}{n!} \right)^2 (2(\psi(n + 1) - \psi(\tau_m/2 + \ell_m + n))v^n
\]

\[
= \sum_{\alpha=0}^{\infty} A_\alpha v^\alpha .
\]

The coefficients \( A_\alpha \) can be written (after transposing the overall factor of \( 1/8 \) to the \( \text{rhs} \) of (A.51) for the two cases of scalar and spin-2 operators as,

\[
A_\alpha = \begin{cases} 
0 & \ell_m = 0 \\
-2P_m \frac{3\Gamma(\tau_m+2\ell_m)}{\Gamma(\tau_m/2+\ell_m)^2} \frac{(\Delta \phi + 2\alpha - 1)\Gamma(\Delta \phi + \alpha - 1)}{\Gamma(\alpha + 1)\Gamma(\Delta \phi)} & \ell_m = 2
\end{cases}
\]

We can thus write (A.54) as,

\[
\sum_{k=0}^{\alpha} q_{\alpha-k, \Delta \phi} d_{\alpha-k, \kappa} C_{\alpha-k} = A_\alpha - b_\alpha \equiv B_\alpha ,
\]

with \( b_\alpha \) given in (A.54). This relation can be inverted in the same spirit as we did for the anomalous dimensions. After inversion the corrections to the OPE coefficients can be written as,

\[
C_n = \Gamma(\Delta \phi - 1)^2 \sum_{m=0}^{n} c_{n,m} B_m ,
\]

where we have defined the coefficients \( B_\alpha \) above and \( c_{n,m} \) is the same coefficient as given in (4.34). Unfortunately to extract a closed form for the coefficients \( C_n \) from the above sum appears difficult. Nevertheless the behavior of the OPE corrections can be inferred from (A.57). In figure (A.1) below we have done a comparative study of the OPE corrections for \( \mathcal{N} = 4 \) SYM [15], when the \( \text{lhs} \) of (4.13) is dominated by a twist-2, spin-2 operator and for twist-2 scalar operators. From the figure we see that at large \( n \), \( C_n \) tend to follow the relation,

\[
C_n = \frac{1}{2q_{\Delta \phi, n}} \partial_n (\bar{q}_{\Delta \phi, n} \gamma_n) .
\]

whereas for small \( n \) there are deviations from the \( \mathcal{N} = 4 \) case. From the inset in figure (A.1) we see that for low lying values of \( n \), \( C_n \) for the twist-2, spin-2 operator exchange becomes negative while those for the \( \mathcal{N} = 4 \) case are positive. \( C_n \) for the scalar exchange case is a constant positive value.

We were unable to extend our calculations to the \( n \gg \ell \gg 1 \) case. The reason is that in order to compute the coefficient \( C_n \) using the methods in this section we would need to know all the
Figure A.1: Plot for $C_n$ for three cases. The blue curve is for $\mathcal{N} = 4$, the red curve for the twist-2, spin-2 operator exchange and the yellow for the twist-2 scalar. We have scaled down the OPE coefficients by a factor $10^8$ in this figure.

Figure A.2: Plot for the numerical estimate of the exponent of $n$ for $\tau_m = 4$ for a range of $\Delta_\phi$.

coefficients $C_0 \cdots C_{n-1}$. This is not possible since we only know the leading order form of $\gamma_n$ in this limit.
A.6 general $\tau_m$ and $\ell_m$

The remaining exponent $\beta$ for $n$ in the expression for the anomalous dimension is determined by doing the following exercise. To start with the anomalous dimension looks like,

$$\gamma(n, \ell) \sim -\frac{n^\beta(\ell + n)^{2-\tau_m}}{\ell(\ell + 2n)}.$$  \hfill (A.59)

In the limit $\ell \gg n \gg 1$, the anomalous dimensions take the form,

$$\gamma(n, \ell) \sim -\frac{n^\beta}{\ell^2}.$$  \hfill (A.60)

We can assume a form of $\beta = a + b\ell_m + c\tau_m$. Putting this back in and calculating for $\tau_m = 2$ and $\ell_m = 2$ (stress tensor) and $\ell_m = 0$ (scalar) for which $\beta = 4$ and 0 respectively, we get the leading term,

$$a + 2b + 2c = 4, \quad a + 2c = 0, \Rightarrow b = 2.$$  \hfill (A.61)

To find the other coefficients $a$ and $c$ we need one more data point. In the plot (A.2) for $\tau_m = 4$ and $\ell_m = 2$, $\beta = 6$. Thus,

$$a + 2b + 4c = 6, \Rightarrow c = 2 \quad \text{and} \quad a = -2.$$  \hfill (A.62)

Thus with $\ell_m = \Delta_m - \tau_m$ for the minimal twist operator,

$$\beta = -2 + 2\Delta_m - \tau_m.$$  \hfill (A.63)

A.7 The $n \gg \ell \gg 1$ case

We will now turn to the $n \gg \ell \gg 1$ case. This is an interesting limit since the impact parameter in the dual gravity side now is small and one could expect to see non-universality corresponding to contributions from higher spin, higher twist exchanges on the lhs of the bootstrap equation which correspond to “stringy” modes. It will not be possible to give a rigorous derivation for the behavior of the anomalous dimensions in this limit. We will make an ansatz for the anomalous dimension and then using a saddle point approximation extract the behavior in this limit. It will turn out that this ansatz correctly captures the $\ell \gg n \gg 1$ case and is in exact agreement with the Eikonal calculation in AdS/CFT. First we will make a change of variables and check that these operators exist.

A.7.1 Existence of double trace operators for $n \gg \ell \gg 1$ limit

Let us demonstrate that for $n \gg \ell \gg 1$, double trace operators exist in large $N$ theories. We first make a change of variables $\tilde{h} = \Delta_\phi + n, \ h = \tilde{h} + \ell$. These are the same variables used in
the AdS/CFT Eikonal approximation \[3\]. The reason for making a change of variables to \(h, \bar{h}\) should be obvious—whenever, \(n, \ell\) are large, irrespective of which is bigger, \(h, \bar{h}\) are both large and \(h \gg \bar{h}\). The \(P^{MFT}\) in terms of these variables is,

\[
P_{MFT} = \frac{2^{7-2(h+\bar{h})} \pi (h + \bar{h} - 2)(h - \bar{h} + 1)}{\Gamma(\Delta_\phi - 1)^2 \Gamma(\Delta_\phi)^2} \frac{\Gamma(\Delta_\phi - 1) \Gamma(h + \Delta_\phi - 2) \Gamma(\bar{h} + \Delta_\phi - 3)}{\Gamma(h - \frac{1}{2}) \Gamma(h - \frac{5}{2}) \Gamma(h + 2 - \Delta_\phi) \Gamma(h + 1 - \Delta_\phi)}.
\] (A.64)

For large \(h\), the conformal block in the crossed channel is,

\[
g_{h,\bar{h}}(v, u) = \frac{2^{2h-1}}{\sqrt{\pi}} h^{1/2} K_0(2h\sqrt{v}) v^h F(\bar{h}, v),
\] (A.65)

where,

\[
F(\bar{h}, v) = \frac{1}{1 - v} 2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v).
\] (A.66)

\(P_{MFT}\) in the limit of large \(h\) can be written as,

\[
P_{MFT} \approx \frac{2^{7-2(h+\bar{h})} \pi \Gamma(\bar{h} - 1) \Gamma(h + \Delta_\phi - 3)}{\Gamma(h - \frac{3}{2}) \Gamma(h + 1 - \Delta_\phi) \Gamma(\Delta_\phi - 1)^2 \Gamma(\Delta_\phi)^2} h^{2\Delta_\phi - 3/2}.
\] (A.67)

Combining this together we find,

\[
\left(\frac{u}{v}\right)^{\Delta_\phi} \sum_{h, \bar{h}} P_{MFT} g_{h, \bar{h}}(v, u) = \frac{4^3 \sqrt{\pi} \Delta_\phi}{\Gamma(\Delta_\phi)^2 \Gamma(\Delta_\phi - 1)^2} \sum_{\text{large } h} h^{2\Delta_\phi - 1} K_0(2h\sqrt{v})
\]

\[
\times \sum_{\bar{h}} \frac{4^{-\bar{h}}(1 - v)^{\Delta_\phi} v^{\bar{h}-\Delta_\phi} \Gamma(\bar{h} - 1) \Gamma(h + \Delta_\phi - 3)}{\Gamma(h - 3/2) \Gamma(h + 1 - \Delta_\phi)} F(\bar{h}, v).
\] (A.68)

Performing the sum (integral) over large \(h\) we get the remaining sum in \(\bar{h}\) as,

\[
A(n, v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_{h, \bar{h}} P_{MFT} g_{h, \bar{h}}(v, u)
\]

\[
= \frac{\Delta_\phi + n}{4^2 \sqrt{\pi} (1 - v)^{\Delta_\phi} v^{\Delta_\phi} \Gamma(\bar{h} - 1) \Gamma(h + \Delta_\phi - 3)} \sum_{\bar{h} = \Delta_\phi} 2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v).
\] (A.69)

We can expand the sum to arbitrarily high orders in \(v\) in mathematica and show that the factor of unity is reproduced on the lhs. This gives evidence of the existence of these \(n \gg \ell \gg 1\) operators.

However this is not enough. These operators in the limit \(n \gg \ell \gg 1\) must also be consistent with the bootstrap equation even at the sub leading order. Next we must argue that the anomalous dimension is small since we are assuming perturbation theory to be able to expand \(v^n\). We will find that for a stress tensor exchange, the anomalous dimension goes like \(n^3/\ell\). So our results will only be valid in the large \(N\) limit with a gap, for some \(n < n_{\text{max}}\). Since there are all powers of \(v\) on the lhs, the question now becomes how to reproduce all the powers. A natural way
would be to argue that the operators above the gap (higher spin modes, “string modes”) will somehow alter the $n$-dependence and allow us to consider any value for $n$ as needed from the $\text{lhs}$. We will initiate a study of this problem.

### A.7.2 Subleading bootstrap equation in terms of $h$ and $\bar{h}$

Since in the $P^{MFT}$ only particular combinations of $h$ and $\bar{h}$ appear, we will assume a general form of the anomalous dimensions in terms of $h$ and $\bar{h}$ to be,

$$
\gamma(h, \bar{h}) \propto h^\alpha \bar{h}^\beta (h + \bar{h})^\chi (h - \bar{h})^\delta,
$$

(A.70)

where $\alpha, \beta, \chi$ and $\delta$ are unknown constants\(^1\). We expect that the form of the anomalous dimension in the limits $\ell \gg n \gg 1$ and $n \gg \ell \gg 1$ case will pertain to special cases of the above expression. Using the Stirling approximations for the $\Gamma$-functions,

$$
\Gamma(a + b) \approx \sqrt{2\pi} \left( \frac{a}{e} \right)^a,
$$

(A.71)

we can write the $P^{MFT}$ as,

$$
P^{MFT} h, \bar{h} \rightarrow \infty \approx \frac{2^{7-2(h+\bar{h})}\pi(h - \bar{h} + 1)(h + \bar{h} - 2)\Gamma(\Delta_\phi^2 - \frac{\gamma}{2})}{\Gamma(\Delta_\phi^2 - 1)^2}.\]

(A.72)

Further for large $h, \bar{h}$ and $z \rightarrow 0$ the conformal blocks in the crossed channel take the form,

$$
g_{h,\bar{h}}(v,u) = 2^{2h-1} \frac{h^{1/2}}{\sqrt{\pi}} K_0(2h\sqrt{z}) \frac{v^\bar{h}}{1 - v} \ 2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v).
$$

(A.73)

Using the anomalous dimensions in terms of $h$ and $\bar{h}$, the $\text{rhs}$ of the bootstrap equation takes the form,

$$
\frac{\sqrt{\pi} z^{\Delta_\phi}}{\Gamma(\Delta_\phi^2 - 1)^2} \int dh \frac{2^{5-2h} h^{2\Delta_\phi - 7 + \beta} v^{\bar{h} - \Delta_\phi}(1 - v)^{\Delta_\phi - 1}}{\Gamma(\Delta_\phi^2 - 1)^2} 2F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v)
$$

$$
\times \int dh h^{2\Delta_\phi - 3 + \alpha (h + \bar{h})^\chi + 1} (h - \bar{h})^\delta + 1 K_0(2h\sqrt{z}).
$$

(A.74)

To sort out the unknown exponents $\alpha, \beta, \chi$ and $\delta$ we will primarily need the $h$ integral which we write out separately for the convenience of the reader.

$$
\int dh h^{2\Delta_\phi - 3 + \alpha (h + \bar{h})^\chi + 1} (h - \bar{h})^\delta + 1 K_0(2h\sqrt{z}).
$$

(A.75)

For large $h$ such that $h\sqrt{z} \gg 1$, we can approximate the Bessel function by,

$$
K_0(2h\sqrt{z}) \approx \sqrt{\pi} \frac{e^{-2h\sqrt{z}}}{2h^{1/2}} z^{1/4}.
$$

(A.76)

---

\(^1\)This form is an assumption. In $P_{MFT}$, $h, \bar{h}, h - \bar{h}, h + \bar{h}$ appear so we will make an ansatz that in the large $h, \bar{h}$ limit, we will get the above form. This form is consistent with the Eikonal result of [3] as we will find.
Plugging this in the $h$ integral, we can write,

$$\frac{\sqrt{\pi}}{2} z^{\Delta_\phi-1/4} \int dh \ h^{2\Delta_\phi-7/2+\alpha}(h+\bar{h})^{\chi+1}(h-\bar{h})^{\delta+1} e^{-2h\sqrt{z}}. \quad (A.77)$$

This integral can be solved in the two limits by taking the appropriate approximations of the quantity $h + \bar{h}$. For the limit $\ell \gg n \gg 1$ case, $h \gg \bar{h}$ and we can write,

$$h + \bar{h} \approx h, \quad (A.78)$$

whereas for $n \gg \ell \gg 1$ case,

$$h + \bar{h} \approx 2\bar{h} \left(1 + \frac{h-\bar{h}}{2\bar{h}}\right) \approx 2\bar{h}. \quad (A.79)$$

The subleading part ($\propto \ell/n$) is neglected in this limit. We will now consider the different limits separately.

**A.7.3 $\ell \gg n \gg 1$**

In this limit $h \gg \bar{h}$ and hence we can write the $h$ integral as (with $\alpha + \chi + \delta = m$),

$$I(h) = \frac{\sqrt{\pi}}{2} z^{\Delta_\phi-1/4} \int dh \ h^{2\Delta_\phi-3/2+m} e^{-2h\sqrt{z}}. \quad (A.80)$$

We can consider the entire function as $e^{g(h)}$ where,

$$g(h) = -2\sqrt{z} + (2\Delta_\phi - 3/2 + m) \log h. \quad (A.81)$$

The saddle is located at,

$$h_0 = \frac{2\Delta_\phi - 3/2 + m}{2\sqrt{z}}. \quad (A.82)$$

Using saddle point approximation, we find that $I(h)$ takes the form,

$$I(h) = \frac{\sqrt{\pi}}{2} z^{\Delta_\phi-1/4} \sqrt{\frac{2\pi}{-g''(h_0)}} e^{g(h_0)} = \frac{\pi}{4} z^{-m/2} \sqrt{2} \left(\frac{2\Delta_\phi - 3/2 + m}{e}\right)^{2\Delta_\phi-3/2+m} \times (2\Delta_\phi - 3/2 + m)^{1/2} e^{-(2\Delta_\phi-3/2+m)}. \quad (A.83)$$

Matching the power of $z$ on both sides we see that $m = \alpha + \chi + \delta = -\tau_m$. The overall coefficient is the same as what we would get if we replace the Bessel function with its exponential form and expand for large $\Delta_\phi$. Thus the $h$ integral takes the form,

$$I(h) = c_{\Delta_\phi} z^{-\tau_m/2}, \quad (A.84)$$
where combining with the overall factors,

\[
c_{\Delta\phi, \tau_m} = \frac{\sqrt{\pi} 2^{1/2+\tau_m-2\Delta\phi} \Gamma(2\Delta\phi - (1/2 + \tau_m))}{\Gamma(\Delta\phi)^2 \Gamma(\Delta\phi - 1)^2}.
\] (A.85)

This is the same overall factor for the \(\ell \gg n \gg 1\) case if we had replaced the function \(K_0(2\ell \sqrt{z})\) with its exponential form and expanded in large \(\Delta\phi\). Thus the \(\bar{h}\) integral becomes,

\[
I(\bar{h}) = \frac{1}{4} c_{\Delta\phi, \tau_m} z^{\tau_m/2} \int \frac{d\bar{h}}{2\pi} \frac{2^{7-2\bar{h}} \bar{h}^{2\Delta\phi-7/2+\beta} \bar{h}^{-\Delta\phi(1-v)} \Delta\phi^{-1} F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v)}{(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v)} \] (A.86)

We can now convert this into the summation form by noting that the factor \(\bar{h}^{2\Delta\phi-7/2+2\bar{h}}\) is the asymptotic form of,

\[
q_{\Delta\phi, n} = \frac{8\Gamma(\Delta\phi + n - 1)^2 \Gamma(n + 2\Delta\phi - 3)}{\Gamma(n + 1) \Gamma(2\Delta\phi + 2n - 3) \Gamma(\Delta\phi)^2 \Gamma(\Delta\phi - 1)^2} \approx \frac{n^{2\Delta\phi-7/2} 2^{7-2\bar{h}}}{\Gamma(\Delta\phi)^2 \Gamma(\Delta\phi - 1)^2}.
\] (A.87)

where for \(n \gg \Delta\phi\) we can take \(\bar{h} = \Delta\phi + n \approx n\). We can further replace \(\bar{h}^\beta\) by \(\gamma_n\) by the \(\gamma_n\) part of \(\gamma(n, \ell)\). Apart from this the other factors in the \(\bar{h}\) integral are exactly the same as for the \(n\) summation. Finally,

\[
\frac{1}{4} \frac{\Gamma(\Delta\phi)^2 \Gamma(\Delta\phi - 1)^2 c_{\Delta\phi, \tau_m} z^{\tau_m/2}}{2\pi} \sum_{n}^{\infty} \gamma_n q_{\Delta\phi, n} v^n (1 - v)^{\Delta\phi - 1} F_1(2\Delta\phi + 2n, v) = lhs.
\] (A.88)

This summation thus reproduces the correct \(n\) dependence of the \(\gamma_n\) functions for this limit as we saw earlier. This argument also fixes the overall sign of the anomalous dimension to be negative.

**A.7.4 \(n \gg \ell \gg 1\)**

In this limit we will neglect the term \((h - \bar{h})/2\bar{h} \gg 1\). Thus the \(h\) integral takes the form,

\[
I(h) = \int dh \frac{h^{2\Delta\phi-2+\alpha+\delta} K_0(2h \sqrt{z})}{2\pi}.
\] (A.89)

Here we have also approximated \(h - \bar{h}\) by \(h\) since we are still in the limit of large \(\ell\). We will further approximate the Bessel function by its exponential form and consider the entire function as \(e^{g(h)}\) where,

\[
g(h) = -2\sqrt{z} + (2\Delta\phi - 5/2 + \alpha + \delta) \log h.
\] (A.90)

Equating \(g'(h) = 0\) gives the location of the saddle \((\alpha + \delta = p)\),

\[
h_0 = \frac{2\Delta\phi - 5/2 + p}{2\sqrt{z}}.
\] (A.91)
Thus,

\[
I(h) = \frac{\sqrt{\pi}}{2} z^{-\Delta_\phi - 1/4} \sqrt{\frac{2\pi}{-g''(h_0)}} e^{g(h_0)} = \frac{\pi}{4} z^{-(p-1)/2} \sqrt{\left(2\Delta_\phi - 5/2 + p\right)^{2\Delta_\phi - 5/2 + p}}
\]

\[
\times (2\Delta_\phi - 5/2 + p)^{1/2} \left(2\Delta_\phi - 5/2 + p\right).
\]  

(A.92)

Thus here \(p = \alpha + \delta = 1 - \tau_m\) and further, the overall coefficient is the same \(c_{\Delta_\phi, \tau_m}\) defined in (A.85). Thus,

\[
I(\bar{h}) = \frac{1}{2} c_{\Delta_\phi, \tau_m} \tau_m^{\tau_m/2} \int d\bar{h} 2^{7+\chi-2\bar{h}} \bar{h}^{2\Delta_\phi - 5/2 + \beta + \chi} \bar{h}^{-\Delta_\phi} (1 - v) \Delta_\phi^{-1} \frac{F_1(\bar{h} - 1, \bar{h} - 1, 2\bar{h} - 2, v)}{v^{\Delta_\phi - 1/2}}.
\]  

(A.93)

Again from our previous discussion we can convert this integral into a summation giving the required behaviour for the limit \(n \gg \ell \gg 1\).

Note that using the two relations,

\[
\alpha + \chi + \delta = -\tau_m, \quad \text{and} \quad \alpha + \delta = 1 - \tau_m, \quad \text{gives} \quad \chi = -1,
\]  

(A.94)

for both the limits. Thus we have partially fixed the form of the anomalous dimension to be,

\[
\gamma(h, \bar{h}) \sim -\frac{(\ell + n)^\alpha n^\beta \ell^\delta}{(\ell + 2n)}.
\]  

(A.95)

From calculating the sub leading terms for \(\ell \gg n \gg 1\) the results from the previous section gives us,

\[
\gamma(h, \bar{h}) \sim -\frac{n^\beta}{\ell^{1-\alpha-\delta}} \left(1 + (\alpha - 2) \frac{n}{\ell}\right),
\]  

(A.96)

so that \(\alpha = 2 - \tau_m\) which further gives \(\delta = -1\). Thus in terms of \(n\) and \(\ell\),

\[
\gamma(n, \ell) \sim -\frac{n^\beta (\ell + n)^{2-\tau_m}}{\ell(\ell + 2n)}.
\]  

(A.97)

The remaining exponent can be obtained (see appendix A.6) by calculating the leading \(n\) dependencies for various twists. It works out to be,

\[
\beta = -2 + 2\ell_m + \tau_m.
\]  

(A.98)

Thus the full expression for the anomalous dimension in the large \(h, \bar{h}\) limit in terms of \(\ell, n\), modulo overall factors, takes the form,

\[
\gamma(n, \ell) \sim -\frac{n^{2+2\ell_m+\tau_m}(\ell + n)^{2-\tau_m}}{\ell(\ell + 2n)}.
\]  

(A.99)

This expression is precisely what emerges from the Eikonal approximation in AdS/CFT [1] for a generic spin exchange. We can see that for the two different limits being considered in our
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paper, it takes the following forms at the leading order,
\[ \gamma(n, \ell) \sim n^{-2+2\ell_m+\tau_m/\ell_m} \], \[ \gamma(n, \ell) \sim n^{2\ell_m-1}/2\ell \]. \hspace{1cm} (A.100)

Note that in the \( n \gg \ell \gg 1 \) case the form just depends on the spin with no dependence on the twist. Moreover, the \( \ell \)-dependence is independent of both \( \tau_m, \ell_m \). In particular, in the limit \( n \gg \ell \gg 1 \) if we have the leading contribution as that coming from a stress tensor\(^2\) for large \( N \) we have
\[ \gamma(n, \ell) \sim -n^3/\ell \left( 1/N^2 + \sum_{\ell_m=2}^{\infty} n^{2\ell_m-4} P^{(m)}_{\ell_m} [1 + O(\frac{\ell}{n})] \right) \] \hspace{1cm} (A.101)

where \( P^{(m)}_{\ell_m} \) are related to the square of the OPE coefficients for massive spin-\( \ell_m \) (\( \ell_m \) even) modes on the \( lhs \) of the bootstrap equation. We are assuming that we have added generic spin and twist on the \( lhs \) so that to produce the appropriate powers of \( u, v \), namely \( u^{\tau_m/2} v^n \log v \) with \( n \) being a natural number\(^3\), we will need to modify the form of the anomalous dimension to what we have indicated above. The leading \( 1/N^2 \) dependence thus will be the leading contribution only if the \( P^{(m)}_{\ell_m} \)'s suppress the contributions from the positive powers of \( n \). In other words for this result to hold there has to be a gap in the spectrum with the contributions from operators above the gap being suppressed. Evidently, this suppression will only work for the \( n \)-dependent operators below the gap. The \( O(\ell/n) \) terms will depend on \( \tau_m, \ell_m \) and are small in this limit. As we keep increasing \( n \sim O(N) \), the assumption that the anomalous dimensions are small will break down (due to the negative sign, one can also be in danger of violating unitarity but this cannot be concluded yet since the anomalous dimension result cannot be trusted when this happens). The interesting question is if adding a single (or a finite number of) higher spin (massive) operator(s) can make the anomalous dimensions small again. The form we have derived above suggests that this is not possible. If we insisted that the operators for all \( n \) have perturbatively small anomalous dimensions, this can only be possible if we resum the contributions from the higher spin modes. The above result seems to suggest that for this to happen one will need an infinite number of higher spin modes since each contribution from the higher spin modes comes with a positive power of \( n \). A more complete analysis of this very important problem is however beyond the scope of this paper (for instance at the level of what we have done we cannot say what the OPE coefficients are for us to be able to re sum the series).

\(^2\)Note that the problem that we allude to in this paragraph does not arise for the \( \ell_m = 0 \) case. This is reminiscent of the discussion in \[4\] where the polarization of the graviton was crucial for the causality arguments.

\(^3\)Of course, for specific values of \( \tau_m \) these sub leading powers of \( u \) will also mix with sub leading \( u \)-powers arising from some leading twist. We are ignoring this possibility.
Appendix B

Details of Chapter 5

B.1 Exact $n$ dependence in $d = 6$

Here we will use the general $d$ expressions discussed in section 5.3. For general $d$, $\gamma_n$ is given by,

$$\gamma_n = \sum_{m=0}^{n} \frac{(4P_m)(-1)^{m+n}n\Gamma(2 + d)\Gamma^2(1 + \frac{d}{2} + m)\Gamma^2(\Delta_\phi)(1 - n)_{m-1}(d - m - n - 2\Delta_\phi)_m}{8\Gamma^2(\Delta_\phi - \frac{d-2}{2})\Gamma^4(1 + \frac{d}{2})\Gamma^2(1 + m)(1 - \frac{d}{2} + \Delta_\phi)_m} \times 3F_2 \left[ \begin{bmatrix} -m, -m, -d + \Delta_\phi \\ -\frac{d}{2} - m, -\frac{d}{2} - m \end{bmatrix}, 1 \right]. \quad (B.1)$$

In $d = 6$ the hypergeometric function can be written as,

$$3F_2 \left[ \begin{bmatrix} -m, -m, -d + \Delta_\phi \\ -\frac{d}{2} - m, -\frac{d}{2} - m \end{bmatrix}, 1 \right] = \sum_{k=0}^{m} \frac{(1 - k + m)^2(2 - k + m)^2(3 - k + m)^2\Gamma(-6 + k + \Delta_\phi)}{(1 + m)^2(2 + m)^2(3 + m)^2k!\Gamma(-6 + \Delta_\phi)} = \frac{36\Gamma(-2 + m + \Delta_\phi)(-2 + 2m + \Delta_\phi)(10(-2 + m)m + \Delta_\phi(-1 + 10m + \Delta_\phi))}{(1 + m)(2 + m)(3 + m)(4 + m)(1 + \Delta_\phi)} \cdot \quad (B.2)$$

To see how to get the above result, we will use the tricks introduced in the appendix of [3]. Note that we can write,

$$(1 - k + m)^2(2 - k + m)^2(3 - k + m)^2 = Ak(k - 1)(k - 2)(k - 3)(k - 4)(k - 5) + Bk(k - 1)(k - 2)(k - 3)(k - 4) + Ck(k - 1)(k - 2) + Dk(k - 1) + Ek(k - 1) + Fk + G. \quad (B.3)$$

where,
\[ A = 1, \ B = 3 - 6m, \ C = 3 + 15m^2, \ D = -2(1 + 2m)(3 + 5m(1 + m)), \]
\[ E = 3(1 + m)^2(6 + 5m(2 + m)), \ F = -3(1 + m)^2(2 + m)^2(3 + 2m), \]
\[ G = (1 + m)^2(2 + m)^2(3 + m)^2. \] (B.4)

So the summation in (B.2) breaks up into seven different sums of the general form shown below,
\[ \sum_{k=0}^{m} k(k-1) \cdots (k-i+1) \frac{\Gamma(x+k)}{k! \Gamma(x)} = \frac{\Gamma(m+x+1)}{(x+i) \Gamma(m-i+1) \Gamma(x)}. \] (B.5)

Once again the above result was derived in [3]. With this and summing up the respective terms, the hypergeometric function simplifies to the form shown in the second line of (B.2).

The main equation (B.1) in \( d = 6 \) has the form,
\[ \gamma_n = \sum_{m=0}^{n} \frac{70(-1)^{n+1} P_m n! \Gamma(-5 + m + n + 2 \Delta \phi) \Gamma(3 - m - \Delta \phi) \Gamma(\Delta \phi) \sin(\Delta \phi \pi)}{\Delta \phi m!(n-m)! \Gamma(-5 + n + 2 \Delta \phi) \pi} \times (-2 + 2m + \Delta \phi)(10m^2 + 10m(-2 + \Delta \phi) + (-1 + \Delta \phi) \Delta \phi). \] (B.6)

To get the above, we have used the reflection formula of gamma functions, \( \Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z) \). To do the sum we write the gamma functions in the numerator as,
\[ \Gamma(3 - m - \Delta \phi)\Gamma(-5 + m + n + 2 \Delta \phi) = \int_0^\infty \int_0^\infty dx dy \, e^{-(x+y)}x^{-6+m+n+2\Delta \phi}y^{2-m-\Delta \phi}. \] (B.7)
Let us write \( f(m) = (-2 + 2m + \Delta_\phi) \left( 10m^2 + 10m(-2 + \Delta_\phi) + (-1 + \Delta_\phi)\Delta_\phi \right) \). Then the sum over \( m \) becomes,

\[
\sum_{m=0}^{n} \frac{(x/y)^m f(m)}{m!(n-m)!} = \frac{(x+y)^n}{y^n(x+y)^3n!} \left( 20n^3x^3 + (x+y)^3(-2 + \Delta_\phi)(-1 + \Delta_\phi)\Delta_\phi \right.
\]
\[+30n^2x^2(x(-2 + \Delta_\phi) + y\Delta_\phi) + 2nx \left( 20x^2 - 3(x+y)(7x + 2y)\Delta_\phi + 6(x+y)^2\Delta_\phi^2 \right) \left) \right) .
\]
(B.8)

The final result can now be obtained by integrating over \( x \) and \( y \). To do this, we change the variables to \( x = r^2\cos^2 \theta \) and \( y = r^2\sin^2 \theta \). Then we get,

\[
\gamma_n = -\frac{70(-1)^nP_m n! \Gamma(\Delta_\phi)\sin(\Delta_\phi \pi)}{\Delta_\phi m!(n-m)!}\int_{0}^{\infty} \int_{0}^{\infty} dx \, dy \, e^{-(x+y)x^{-6+n+2\Delta_\phi}y^{-2-\Delta_\phi}} \sum_{m=0}^{n} \frac{(x/y)^m f(m)}{m!(n-m)!}
\]
\[= -\frac{70}{\Delta_\phi^2} P_m \left( 20n^6 + 60(-5 + 2\Delta_\phi) n^5 + 10 \left( 170 - 132\Delta_\phi + 27\Delta_\phi^2 \right) n^4 \right.
\]
\[+ 20(-5 + 2\Delta_\phi) \left( 45 - 32\Delta_\phi + 7\Delta_\phi^2 \right) n^3 + 2 \left( 2740 - 3780\Delta_\phi + 2121\Delta_\phi^2 - 582\Delta_\phi^3 + 66\Delta_\phi^4 \right) n^2
\]
\[+ 4(-5 + 2\Delta_\phi) \left( 120 - 140\Delta_\phi + 73\Delta_\phi^2 - 21\Delta_\phi^3 + 3\Delta_\phi^4 \right) n + (-2 + \Delta_\phi)^2 \left( -1 + \Delta_\phi \right)^2 \Delta_\phi^2 \left) \left) \right) .
\]
(B.9)

This is the result for \( \gamma_n \) in \( d = 6 \). Note that we simply extended the tricks used for \( d = 4 \) in the appendix of [3]. The same can be done to evaluate \( \gamma_n \) for any even \( d > 2 \).

The figure B.1 shows the range of \( \Delta_\phi \), in \( d = 4 \) and \( 6 \) for which \( \gamma_{n=1} \) takes positive values. Interestingly, for both the cases positive values of \( \gamma_1 \) are over those values of \( \Delta_\phi \) which violate the respective unitarity bounds. In fact this behavior of the anomalous dimensions is true in any dimension. We found that for \( n \geq d - 1 \), \( \gamma_n \) was negative for any \( \Delta_\phi > 0 \).
Appendix C

Details of Chapter 6

C.1 Determination of $T_d(p,q)$ in $4 - \epsilon$ dimensions

We will now proceed with the determination of $T_d(p,q)$ in $4 - \epsilon$ dimensions as follows: This will be largely based on the analysis in appendix A of [6] except that there the result was presented in 4 dimensions. We will also stick to the sign conventions as in appendix A of [6] but in the main text the sign of $q$ will be reversed to be compatible with the direction of momenta in figure (6.1). The momentum space representation of the three point vertex can be written as,

$$\langle T\mathcal{O}(R)\phi(r)\phi(0) \rangle = \int d^{4-\epsilon}r d^{4-\epsilon}R e^{ipr + iqR} \frac{e^{ipr + iqR}}{r^2 \Delta - d |R - r|^d},$$

(C.1)

where $\Delta$ is the conformal dimension of $\phi$ and $d$ is the conformal dimension of the operator $\mathcal{O}$. Using the Schwinger parametrization we can write,

$$\frac{1}{a^s} = \frac{1}{\Gamma(s)} \int_0^\infty dx x^{s-1} e^{-ax}.$$  \hspace{1cm} (C.2)

Thus the above three point function in the momentum space becomes,

$$\langle T\mathcal{O}(R)\phi(r)\phi(0) \rangle = \text{const.} \int dx \, dy \, dz (xy)^{d/2-1} \Delta^{-d/2-1} \int d^{4-\epsilon}r d^{4-\epsilon}R \, e^{f(x,y,z,r,R)},$$

(C.3)

where the function $f$ is given by,

$$f(x, y, z, r, R) = ipr + iqR - xR^2 - y(R - r)^2 - zr^2.$$  \hspace{1cm} (C.4)

What we do in the beginning is complete the squares for the function $f$ and isolate the parts depending solely on $r$ and $R$. After completing the squares, we get,

$$f(x, y, z, r, R) = -A^2 - B^2 - f_1(x, y, z),$$

(C.5)
where again the functions are given by,

\[
A = \sqrt{y + zr} + \frac{2yR - ip}{2\sqrt{y + z}} , \\
B = \sqrt{\frac{xy + yz + xz}{y + z}} R + i \frac{(p + q)y + qz}{\sqrt{(y + z)(xy + yz + xz)}} .
\]

(C.6)

We can first give a shift to the functions \( A \) and \( B \) and then carry out the integrals over the shifted variables by noticing that the integral measure is given by,

\[
\int d^{4-\epsilon} r \to 4\pi \int r^{3-\epsilon} dr .
\]

(C.7)

Thus the integrals over \( r \) and \( R \) contribute to,

\[
\int d^{1-\epsilon} r d^{1-\epsilon} R e^{-(A^2 + B^2)} = 4\pi^2 \Gamma(2 - \epsilon/2)^2 (xy + yz + xz)^{-2+\epsilon/2} ,
\]

and thus,

\[
\text{F.T.}(TO(R)\phi(r)\phi(0)) = \int dx dy dz \frac{(xy)^{d/2 - 1}z^{\Delta - d/2 - 1}}{(xy + yz + xz)^{2-\epsilon/2}} e^{-f(x,y,z)} .
\]

(C.9)

Replacing \((x, y, z) \to 1/4(1/x, 1/y, 1/z)\) gives,

\[
\text{F.T.}(TO(R)\phi(r)\phi(0)) = \int dx dy dz \frac{(xy)^{1-d/2-\epsilon/2}z^{d/2+1-\Delta-\epsilon/2}}{(x + y + z)^{2-\epsilon/2}} e^{-\frac{x^2 + y^2 + z^2}{(x+y+z)^2}} .
\]

(C.10)

We will now change to the coordinates \( z = \nu \rho, x = \rho \lambda \) and \( y = \rho (1 - \lambda) \) and the integration limits are \( 0 < \nu, \rho < \infty \) and \( 0 < \lambda < 1 \) respectively. Thus the function in the argument of the exponential changes from,

\[
f(\rho, \nu, \lambda) = -\frac{\rho}{1 + \nu} (g(\lambda) + \nu (h(\lambda))) , \quad g(\lambda) = q^2 \lambda (1 - \lambda) , \quad h(\lambda) = p^2 \lambda + (1 - \lambda)(p + q)^2 .
\]

(C.11)

In these coordinates the integral takes the form,

\[
\int d\rho d\nu d\lambda \rho^2 (\rho^2 \lambda (1 - \lambda))^{1-\epsilon/2-d/2} (\nu \rho)^{d/2+3-\Delta-\epsilon/2} e^{-f(\nu, \lambda)\rho} ,
\]

(C.12)

where \( f(\nu, \lambda) \) is the remaining part of the argument in the exponential apart from \( \rho \). Performing the integral over \( \rho \) we get,

\[
\int_0^1 d\lambda (\lambda (1 - \lambda))^{1-d/2-\epsilon/2} \int_0^\infty d\nu \frac{\nu^{d/2+1-\Delta-\epsilon/2}(1 + \nu)^{2-\epsilon/2-d/2-\Delta}}{[g(\lambda) + \nu h(\lambda)]^{4-\epsilon-d/2-\Delta}} .
\]

(C.13)

Finally performing the integral over \( \nu \) we get,

\[
\text{F.T.}(TO(R)\phi(r)\phi(0)) = N_d T_{4-\epsilon-d}(p, q) + L_d T_d(p, q) q^{2d-4+\epsilon} ,
\]

(C.14)
where,

\[ N_d = \frac{\Gamma(\Delta - d/2)\Gamma(d - 2 + \epsilon/2)}{\Gamma(\Delta + d/2 - 2 + \epsilon/2)} , \quad L_d = \frac{\Gamma(2 + d/2 - \Delta - \epsilon/2)\Gamma(2 - d - \epsilon/2)}{\Gamma(4 - d/2 - \Delta - \epsilon)} \]  

(C.15)

\[ T_d(p, q) = \int_0^1 d\lambda \frac{(\lambda(1 - \lambda))^{d/2-1}}{[\lambda p^2 + (1 - \lambda)(p + q)^2]^{2+d/2-\Delta-\epsilon/2}} \times _2F_1 \left[ 2 + d/2 - \Delta - \epsilon/2, \Delta + d/2 - 2 + \epsilon/2, d - 1 + \epsilon/2, \frac{q^2\lambda(1 - \lambda)}{\lambda p^2 + (1 - \lambda)(p + q)^2} \right]. \]  

(C.16)

In the expression for the three point functions we will take the discontinuous part which amounts to taking the part proportional to \( q^{2d-4+\epsilon} \).

### C.2 Mixing coefficients

We will derive the mixing factors for the channels below. The full scattering amplitude is given by

\[ A^{(s)}_{ijkl} = a_0 T^0(q,p) \text{Im} D^0(q) T^0(q,p') \delta_{ij} \delta_{kl} + a_2 T^2(q,p) \text{Im} D^2(q) T^2(q,p') (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{2}{n} \delta_{ij} \delta_{kl} \]  

(C.17)

where the labels \( I = 0, 2 \) denote the contribution of the exchanged scalars \( O^{(0)} \) and \( O^{(2)} \) in the theory. The \( t \)-channel is obtained by switching \( j \leftrightarrow k \) and the \( u \)-channel is obtained by switching \( j \leftrightarrow l \) respectively. Thus

\[ A^{(t)}_{ijkl} = a_0 T^0(q,p) \text{Im} D^0(q) T^0(q,p') \delta_{ik} \delta_{jl} + a_2 T^2(q,p) \text{Im} D^2(q) T^2(q,p') (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}) - \frac{2}{n} \delta_{ik} \delta_{jl} \],

\[ A^{(u)}_{ijkl} = a_0 T^0(q,p) \text{Im} D^0(q) T^0(q,p') \delta_{il} \delta_{jk} + a_2 T^2(q,p) \text{Im} D^2(q) T^2(q,p') (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) - \frac{2}{n} \delta_{il} \delta_{jk} \].  

(C.18)

Thus multiplying the total amplitude \( A_{ijkl} = A^{(s)}_{ijkl} + A^{(t)}_{ijkl} + A^{(u)}_{ijkl} \) by \( \delta^{ij} \delta^{kl} \) we have,

\[ A_{ijkl} \delta^{ij} \delta^{kl} = n^2 \left[ a_0 \left( 1 + \frac{2}{n} \right) T^0 \text{Im} D^0(q) T^0 + \frac{2n^2 + 2n - 4}{n^2} a_2 T^2 \text{Im} D^2(q) T^2 \right], \]

\[ A_{ijkl} \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \frac{2}{n} \delta^{ij} \delta^{kl} \right) = (2n^2 + 2n - 4) \left[ a_2 T^2 \text{Im} D^2(q) T^2 + a_0 T^0 \text{Im} D^0(q) T^0 \right. \]

\[ + \frac{n-2}{n} a_2 T^2 \text{Im} D^2(q) T^2 \].  

(C.19)
In the limit where the $t$ and the $u$ channel contributes, the $\delta^i$ dependent factors drop out from both $T^0 \text{Im} D^0(q) T^0$ and $T^2 \text{Im} D^2(q) T^2$. Thus we can write the above expression as,

$$A_{ijkl} = \left[ a_I + \sum_{J=0,2} c_{IJ} a_J \right] f(v). \quad (C.20)$$

Comparing with the above formula for $A_{ijkl}$ we get,

$$c_{00} = \frac{2}{n}, \quad c_{02} = \frac{2n^2 + 2n - 4}{n^2}, \quad c_{20} = 1, \quad c_{22} = \frac{n - 2}{n}. \quad (C.21)$$

### C.3 Evaluation of some generic integrals

In the evaluation above we will be facing the integrals of two generic types. We digress a little to show the evaluation of these integrals. First note that the four point amplitudes in the momentum space involve terms of the form,

$$A(q, p, p', p'') \sim q^a p^b p'^c. \quad (C.22)$$

The Fourier transform of these power law behavior in the regime of interest is not very difficult to evaluate.

$$\int_0^{R^{-1}} d^{4-\epsilon} q \int_{R^{-1}}^{r'^{-1}} \frac{dp'}{p'} p'^{-\epsilon} \left( \int_{p'}^{1/r'} + \int_{p'}^{1/r''} \right) \frac{dp}{p} p^{-\epsilon} q^a p^b p'^c. \quad (C.23)$$

Using the Euclidean representation for the integral over $q$ we can show that,

$$\int_0^{R^{-1}} d^{4-\epsilon} q \ q^a = \frac{1}{4 + a - \epsilon} R^{4-a}. \quad (C.24)$$

For the remaining part, the integral over $p$ and $p'$ gives,

$$\frac{1}{b - \epsilon} \int_{R^{-1}}^{r'^{-1}} \frac{dp'}{p'} p'^{-\epsilon} \left( r'^{b-\epsilon} + r'^{c-\epsilon} - 2p'^{b-c} \right) = \frac{1}{b - \epsilon} \left[ \frac{1}{c - \epsilon} \left( r'^{b-\epsilon} + r'^{c-\epsilon} - R'^{-c} \right) \right. \left. - \frac{2}{b + c - 2\epsilon} \left( r'^{2c-b-c} - R'^{2c-b-c} \right) \right]. \quad (C.25)$$

Combining this together we have for $b \neq \epsilon$,

$$S_{a,b,c}(r, r', R) = \int_0^{R^{-1}} d^{4-\epsilon} q \int_{R^{-1}}^{r'^{-1}} \frac{dp'}{p'} p'^{-\epsilon} \left( \int_{p'}^{1/r'} + \int_{p'}^{1/r''} \right) \frac{dp}{p} p^{-\epsilon} q^a p^b p'^c$$

$$= \frac{R'^{-4-a}}{(b - \epsilon)(4 + a - \epsilon)} \left[ \frac{1}{c - \epsilon} \left( r'^{b-\epsilon} + r'^{c-\epsilon} - R'^{c-\epsilon} \right) \right. \left. - \frac{2}{b + c - 2\epsilon} \left( r'^{2c-b-c} - R'^{2c-b-c} \right) \right]. \quad (C.26)$$
When \( b = \epsilon \), the integral over \( p \) and \( p' \) gives,

\[
T_{a,c}(r, r', R) = \int_0^{R^{-1}} d^{d-\epsilon} q \int_{R^{-1}}^{r^{-1}} \frac{dp' - \epsilon}{p'} \left( \int_{p'}^{1/r'} \frac{dp}{p} p^\epsilon q^a p' p'^c \right)
\]

\[
= \frac{R^{4-a}}{(4 + a - \epsilon)(c - \epsilon)} \left[ \log \left( \frac{1}{r'} \right) (r'^{c'-c} - R^{c'-c}) - 2(R^{c'-c} \log R - r'^{c'-c} \log r') \right] \quad (C.27)
\]

### C.4 Generic three point function

To calculate the three point functions of the form \( \langle \phi_i(r) \phi_j(0)(J_{ij})_{kl}(R) \rangle \) we will need to evaluate some generic integrals in the momentum space of the form,

\[
F.T.\langle T \mathcal{O}_{kl}(R) \phi_i(r) \phi_j(0) \rangle = \mathcal{I}_{ijkl} \int d^{d-\epsilon} r d^{d-\epsilon} R \frac{e^{i(p + q R)}}{(R - r)^a R^{b+c}} , \quad (C.28)
\]

where \( \mathcal{I}_{ijkl} \) is the associated \( O(N) \) tensor structure (warning: here \( a \) is not \( 4 - \epsilon \)). Using the Schwinger parametrization of the propagators as in (C.2), we can get an integral similar to (C.9) (modulo the tensor structure)

\[
F.T.\langle T \mathcal{O}_{kl}(R) \phi_i(r) \phi_j(0) \rangle = \text{const} \cdot \int dx dy dz \frac{x^{a/2-1}y^{b/2-1}z^{c/2-1}}{(xy + yz + xz)^{2-\epsilon/2}} f(x, y, z) . \quad (C.29)
\]

where \( f(x, y, z) \) has the exact same structure as in (C.6). We can now do the analogous change of variables from \((x, y, z) \rightarrow 1/(1/x, 1/y, 1/z)\) to get,

\[
F.T.\langle T \mathcal{O}_{kl}(R) \phi_i(r) \phi_j(0) \rangle = \text{const} \cdot \int dx dy dz \frac{x^{1-a/2-\epsilon/2}y^{1-b/2-\epsilon/2}z^{1-c/2-\epsilon/2}}{(x + y + z)^{2-\epsilon/2}} e^{-f(1/x, 1/y, 1/z)} . \quad (C.30)
\]

Changing the variables to,

\[
x = \rho \lambda , \quad y = \rho (1 - \lambda) , \quad \text{and} \quad z = \rho \nu , \quad (C.31)
\]

we can put the integral above in the form,

\[
F.T.\langle T \mathcal{O}_{kl}(R) \phi_i(r) \phi_j(0) \rangle = \text{const} \cdot \int_0^\infty \rho^{3-a+b+c-\epsilon} e^{-f(\lambda, \rho)} d\lambda d\nu \lambda^{1-\epsilon/2} (1 - \lambda)^{1-\epsilon/2 - \frac{\nu}{2}} \frac{\nu^{1-\epsilon/2 - \frac{\nu}{2}}}{(1 + \nu)^{2-\frac{\nu}{2}}} ,
\]

\[
= \text{const} \cdot \int_0^1 d\lambda \lambda^{1-\frac{\nu}{2}} (1 - \lambda)^{1-\frac{\nu}{2} - \frac{\nu}{2}} \int_0^\infty d\nu \frac{\nu^{1-\frac{\nu}{2}} (1 + \nu)^{2-\frac{\nu}{2} - a+b+c}}{[g(\lambda) + \nu h(\lambda)]^{4-\epsilon-\frac{a+b+c}{2}}} ,
\]

\[
= \text{const} \cdot [N_{a,b,c} T^1_{a,b,c}(p, q) + L_{a,b,c} T^2_{a,b,c}(p, q)] . \quad (C.32)
\]
where \( g(\lambda) \) and \( h(\lambda) \) are defined in (C.11) and the two constants take the form,

\[
N_{a,b,c} = \frac{\Gamma(\frac{2}{3})\Gamma(-2 + \frac{5}{2} + \frac{a+b}{2})}{\Gamma(-2 + \frac{5}{2} + \frac{a+b+c+\epsilon}{2})}, \quad L_{a,b,c} = \frac{\Gamma(2 - \frac{5}{2} - \frac{a+b}{2})\Gamma(2 - \frac{5}{2} - \frac{5}{2})}{\Gamma(4 - \epsilon - \frac{a+b+c}{2})}.
\]

The functions \( T^1 \) and \( T^2 \) are given by,

\[
T^1_{a,b,c} = \int d\lambda \frac{\lambda^{\frac{1}{2} - \frac{2}{3} - \frac{1}{2} - \frac{5}{2} - \frac{a+b}{2}}}{h(\lambda)^{\frac{1}{2} - \frac{a+b+c+\epsilon}{2}}} _2F_1 \left[ \frac{c}{2}, 4 - \epsilon - \frac{a + b + c}{2}, 3 - \epsilon - \frac{a + b}{2}, \frac{g(\lambda)}{h(\lambda)} \right],
\]

\[
T^2_{a,b,c} = \int d\lambda \frac{\lambda^{\frac{1}{2} - \frac{2}{3} - \frac{1}{2} - \frac{5}{2} - \frac{a+b}{2}}}{h(\lambda)^{\frac{1}{2} - \frac{a+b+c+\epsilon}{2}}} _2F_1 \left[ 2 - \frac{c + \epsilon}{2}, -2 + \frac{a + b + c + \epsilon}{2}, -1 + \frac{a + b + \epsilon}{2}, \frac{g(\lambda)}{h(\lambda)} \right],
\]

where \( g(\lambda) \) and \( h(\lambda) \) are given in (C.11).

### C.5 Four point Green function of the mixed correlator

We quote the relevant details for the construction of the four point Green function for the mixed correlator in the position space.

#### C.5.1 Unitary Green function

The way to schematically write down the four point function in momentum space, is,

\[
F.T \{ \langle T \phi_i(r) \phi_j(0) \phi_k(R) \phi_l(R+r') \rangle \} = A_I(q, p, p') = \int_s^\infty \frac{ds'}{s'} T^2_{\phi \phi \phi \phi}(s', p) \text{Im}D(s') T^2_{\phi(s^2 \phi \phi)}(s', p'),
\]

where the associated three point functions relevant for the leading discontinuous part of the amplitude is given by \( T^2_{a,b,c} \) functions of the appendix (C.4). We have also suppressed the indices associated with the four point amplitude for convenience. Later we will use superscripts \( s, t, u \) to denote the contributions due to the individual channels. Since we have already defined the generic three point functions for scalars, we can refer to (C.34) of (C.4) for this section. We will need two different kinds of three point functions for our purposes. These are defined as \( F_1 = T^2_{d,d,2\Delta - d} \) and \( G_1 = T^2_{d+\Delta_3 - \Delta, d+\Delta - \Delta_3, \Delta + \Delta_3 - d} \). Thus the four point amplitude in the momentum space becomes,

\[
A_I(q, p, p') = \int_s^\infty \frac{ds'}{s'} F_1(s', p) \text{Im}D(s') G_1(s', p').
\]

This four point function should now be solved in the regime \( p \gg p' \gg q \) for the cases when \( s \gg (v, w) \) and \( s \ll (v, w) \) respectively. We will now proceed to evaluate the limits for the individual functions \( F_1 \) and \( G_1 \) separately. For the function \( F_1 \), we know the limits from our previous calculations. Barring the overall factors, the expression for \( F_1 \) in the two limits take
where the terms in (C.39),

\[
F_1(s, v) = \frac{1}{v^2} \begin{cases} 
  f_1 v^{-d/2+\Delta+\epsilon/2} & : s \ll v \\
  f_2 s^{2-d-2-\Delta-\epsilon/2} v^{-2+2\Delta+\epsilon} & : s \gg v 
\end{cases}
\]

where the coefficients \(f_1\) and \(f_2\) can be written as,

\[
f_1 = \frac{\Gamma(d/2)^2}{\Gamma(d)} , \quad f_2 = \frac{\Gamma(2-\Delta - \frac{\epsilon}{2})^2 \Gamma(d-1 + \frac{\epsilon}{2})}{\Gamma(1+d/2-\Delta) \Gamma(2+d/2-\Delta-\frac{\epsilon}{2})} \text{Re}[(-1)^{\frac{1}{2}}(-d-2\Delta-\epsilon)].
\]

And for the other function we have,

\[
G_1(s, v) = \frac{1}{v^2} \begin{cases} 
  g_1 v^\frac{\Delta+\Delta_3-d+\epsilon}{2} & : s \ll v \\
  g_2^{(0)} s^{-2-d/2+\Delta+\Delta_3+\epsilon} v^2 + g_2^{(1)} s^{2-d/2-\Delta+\Delta_3+\epsilon} v^{-2+\Delta+\Delta_3+\epsilon} & : s \gg v 
\end{cases}
\]

where the coefficients are,

\[
g_1 = \frac{\Gamma((d+\Delta-\Delta_3)/2) \Gamma((d+\Delta_3-\Delta)/2)}{\Gamma(d)}, \\
g_2^{(0)} = \frac{\Gamma(d-1+\frac{\epsilon}{2}) \Gamma(-2+\Delta+\frac{\epsilon}{2}) \Gamma(-2+\Delta_3+\frac{\epsilon}{2})}{\Gamma(-2+d+\Delta+\Delta_3+\epsilon)} \text{Re}[(-1)^{\frac{1}{2}}(-d+\Delta+\Delta_3+\epsilon)], \\
g_2^{(1)} = \frac{\Gamma(2-\Delta-\frac{\epsilon}{2}) \Gamma(2-\Delta_3-\frac{\epsilon}{2}) \Gamma(d-1+\frac{\epsilon}{2})}{\Gamma(1+d-\Delta-\Delta_3-\epsilon)} \text{Re}[(-1)^{\frac{1}{2}}(-d-\Delta-\Delta_3-\epsilon)].
\]

In addition to the above information, we also have for the exchange,

\[
\text{Im} D(s) = \text{const.} \, s^{d-a/2}.
\]

We can thus break up the integral over \(s'\) into regimes

\[
A(g, p, p') = \int_s^w ds' \left( \cdots \right) + \int_s^v ds' \left( \cdots \right) + \int_v^\infty ds' \left( \cdots \right),
\]

where the terms in \(\left( \cdots \right)\) denote the function \(F_1 \times \text{Im} D(s) \times G_1\) in various limits and \(v = p^2\), \(s = q^2\) and \(w = p'^2\) respectively. We can spell out the different individual limits for the convenience of the reader.

- \(s \ll v, w\). In this case, we find that,

\[
F_1(s, v) \text{ Im} D(s) \ G_1(s, w) = \frac{1}{v^2 w^2} f_1 g_1 v^{-d/2+\Delta+\epsilon/2} w^{\Delta+\Delta_3-d+\epsilon} s^{d-a/2}.
\]

- \(w \ll s \ll v\). In this case, we find that,

\[
F_1(s, v) \text{ Im} D(s) \ G_1(s, w) = \frac{1}{v^2 w^2} f_1 v^{-d/2+\Delta+\epsilon/2} s^{d-a/2} g_2^{(0)} s^{2-d/2+\Delta+\Delta_3+\epsilon} w^2 + g_2^{(1)} s^{2-d/2-\Delta+\Delta_3+\epsilon} w^{-2+\Delta+\Delta_3+\epsilon}.
\]
• \( s \gg v, w \). In this case, we find,

\[
F_1(s, v) \text{ ImD}(s) \ G_1(s, w) = \frac{1}{v^2 w^2} f_2 s^{2+d/2-a/2-\Delta-\epsilon/2} v^{-2+2\Delta+\epsilon} [g_0^2 s^{-2-d/2+\Delta+\epsilon} + g_1^2 s^{2-d/2+\Delta+\epsilon}] .
\]

(C.45)

We are now supposed to complete the integral over \( s' \) to get the four point amplitude \( A_{ijkl}(q, p, p') \). Modulo the overall coefficients which are present and the factors due to the \( s' \) integral, we can write this as a function of four different terms

\[
A_{I}^{(s)}(q, p, p') = \frac{a_I}{v^2 w^2} [c_1 s^{-d/2} v^{-d/2+\Delta+\epsilon/2} w^{(\Delta+\Delta_3-d-a)} + c_2 v^{-d/2+\Delta+\epsilon/2} w^{(\Delta+\Delta_3-d-a+\epsilon/2)} + c_3 v^{-2-a/2+(3\Delta+\Delta_3)/2+\epsilon} w^2 + c_4 v^{-2-a/2+(\Delta-\Delta_3)/2} w^{-2+\Delta+\Delta_3+\epsilon}] ,
\]

(C.46)

where the coefficients in the above expression are given by,

\[
c_1 = -\frac{f_1g_1}{d-a/2} ,
\]

\[
c_2 = \frac{f_1g_1}{d-a/2} - 2 \left[ \frac{f_1g_2(0)}{d-4-a+\epsilon+\Delta+\Delta_3} + \frac{f_1g_2(1)}{d+4-a-\epsilon-\Delta-\Delta_3} \right] ,
\]

(C.47)

\[
c_3 = \frac{2f_1g_2(0)}{d-4-a+\epsilon+\Delta+\Delta_3} - \frac{2f_2g_2(0)}{\Delta_3-\Delta-a} ,
\]

\[
c_4 = \frac{2f_1g_2(1)}{d+4-a-\epsilon-\Delta-\Delta_3} - \frac{2f_2g_2(1)}{4-a/2-\epsilon-(\Delta_3+3\Delta)/2} .
\]

We have omitted the overall tensor structure on the rhs which is to say that these functions will depend on the kind of exchange we have and that will reflect through the individual coefficients \( c_1^I \) etc. appearing above. The superscript \( (s) \) denotes that the above expressions are for the \( s \)-channel decomposition of the amplitude. The \( t \) and the \( u \) channel will only have the last limit \( v, w \) and hence only the expression (C.45) in the decomposition albeit with some different coefficients. Thus for the \( t \) and the \( u \) channel there is a change of the variables for the three point functions. For the \( \langle \phi \phi \mathcal{O} \rangle \) correlator notice that, the function \( T_3^2(p, q) \rightarrow T_3^2(p-q, p-p') \) and similarly for the other three point function \( \langle \phi (\phi^2 \mathcal{O}) \rangle \). Notice also that the exchange momentum is now \( s = (p-p')^2 \) and for the regime \( P \gg p' \gg q, s \gg v \). Thus for the \( t \) channel only one regime is relevant \( v, w \). So it will suffice to calculate the three point functions in this regime. The respective three point functions are given by,

\[
\langle \phi \phi \mathcal{O} \rangle = \frac{1}{v^2 w^2} h_1 s^{2-d/2-\Delta-\epsilon/2} (vw)^{\Delta+\epsilon/2} : s \gg v, w ,
\]

\[
\langle \phi (\phi^2 \mathcal{O}) \rangle = h_2 s^{-2-d/2-\Delta+\epsilon/2} v^{\Delta-\epsilon/2} w^{-2+\Delta_3+\epsilon} + h_3 s^{-2-d/2+\Delta+\epsilon/2} : s \gg v, w .
\]

(C.48)
where the explicit form of the coefficients $h_1$, $h_2$ and $h_3$ are given by,

$$h_1 = \frac{\Gamma(2 - \Delta - \frac{d}{2})^2 \Gamma(d - 1 + \frac{d}{2})}{\Gamma(1 + \frac{d}{2} - \Delta) \Gamma(2 + \frac{d}{2} - \Delta - \frac{d}{2})} \text{Re}((-1)^{-\frac{1}{2}(d+2\Delta+\epsilon)})$$

$$h_2 = \frac{\Gamma(2 - \Delta - \frac{d}{2}) \Gamma(2 - \Delta_3 - \frac{d}{2}) \Gamma(d - 1 + \frac{d}{2})}{\Gamma(1 + \frac{d-\Delta-\Delta_3}{2}) \Gamma(2 + \frac{d-\Delta-\Delta_3}{2} - \epsilon)} \text{Re}((-1)^{-\frac{1}{2}(d+\Delta+\Delta_3+\epsilon)})$$

$$h_3 = \frac{\Gamma(d - 1 + \frac{d}{2}) \Gamma(-2 + \Delta + \frac{d}{2}) \Gamma(-2 + \Delta_3 + \frac{d}{2})}{\Gamma(-2 + \frac{d+\Delta+\Delta_3}{2}) \Gamma(-3 + \epsilon + \frac{d+\Delta+\Delta_3}{2})} \text{Re}((-1)^{-\frac{1}{2}(-d+\Delta+\Delta_3+\epsilon)})$$

Similarly, for the $u$ channel, the exchange momentum is $s = (p + p' - q)^2$ which for our regime of interest is again $s \geq v$. Hence the above limits of the three point functions apply in this channel as well. The full four point function can be written as,

$$A_I^{(l)}(s, v, w) = \langle \phi_i \phi_j O \rangle \text{ Im} D(s) \langle \phi_k (\phi^2 \phi_l) O \rangle .$$

Integrating over the variable $s$ in the regime $s \gg v \gg w$ one can see that,

$$A_I^{(l)}(v, w) = \int_v^\infty \frac{ds}{s'} A_I^{(l)}(s', v, w) = \frac{2}{v^2 w^2} \left[ \frac{h_1 h_2}{8 + a + 3\Delta + \Delta_3 + 2\epsilon} v^{2 - \frac{a}{2} + \Delta + \Delta_3} w^{-2 + \Delta + \Delta_3 + \epsilon} + \frac{h_1 h_3}{a + \Delta - \Delta_3} v^{-\frac{a}{2} + \Delta + \Delta_3} w^{\Delta + \frac{\Delta_3}{2}} \right] .$$

We are now in a position to address the question about whether this mixed correlator can determine the anomalous dimension for the operators $\phi_i$ and the composite operator $\phi^2 \phi_i$ given the information about the anomalous dimensions for the exchange operators. The key idea will be now to convert the sum of the contributions from the three channels into position space representation and impose the operator algebra which we now proceed to do in the next section.

### C.5.2 Position space representation

The momentum space amplitude can now be converted into the position space through the Fourier transform so that we can impose the operator algebra (OPE) on that. We will consider individual terms in individual channels for clarity.

#### C.5.2.1 $s$ channel

Consider the $s$ channel first in (C.46). In terms of $v = p^2$, $w = p'^2$ and $s = q^2$ this becomes,

$$A_s^I(q, p, p') = \frac{a_I}{p^4 p'^4} \left[ c_{1q}^{2d-a} p^{-d+2\Delta+\epsilon} p'^{\Delta+\Delta_3-d+\epsilon} + c_{2p}^{d+2\Delta+\epsilon} p'^{\Delta+\Delta_3+d-a+\epsilon} + c_{3}^{4-d-a-\Delta_3+2\epsilon} p'^4 + c_{4}^{4-a-\Delta_3+\Delta_3+2\Delta+2(\Delta+\Delta_3)+2\epsilon} \right] .$$

(C.52)
The label $i$ indicates what scalar ($i = 0, 2$) is exchanged in this channel. The Fourier transform of the above function is,

$$G_I(r, r', R) = \int d^{4-\epsilon}p \ d^{4-\epsilon}p' \ d^{4-\epsilon}q \ e^{i(p r + p' r' + q R)} A_I(q, p, p').$$  \hfill (C.53)

In the regime we are interested, we can simplify the measure further by writing,

$$\int \frac{1}{p \ p'} d^{4-\epsilon}p \ d^{4-\epsilon}p' \ d^{4-\epsilon}q \ e^{i(p r + p' r' + q R)}(\cdots) \equiv \int_0^{R-1} d^3q \int_{R-1}^{r-1} \frac{dp'}{p'} \int_{p'}^{r'-1} \frac{dp}{p} \ p^{-\epsilon}(\cdots)$$  \hfill (C.54)

We refer the reader to the appendix C.1 for the details of the Fourier transforms and quote here the final results. We are interested in the leading power law dependence of the position space Green functions after the matching with the OPE part. Now we are in a position to match various coefficients. First notice that in the position space representation, two terms appear repeatedly. These are,

$$r'^a - 3\Delta - \Delta^3 \quad \text{and} \quad R^a - 3\Delta - \Delta^3 \quad \text{(C.55)}$$

But for the regime of interest, we have chosen $r \sim r' \ll R$ and even at the leading order we know that $a - 3\Delta - \Delta^3 = -2 + O(\epsilon)$. Thus the dominant contribution in the specific regime will come from $r'^a - 3\Delta - \Delta^3$ term in the position space representations. We can thus compare the coefficients of this particular term in the position space representation of the Green function.

The coefficients for this term is,

$$- \frac{a_l}{a(a-3\Delta-\Delta^3)} \left[ \frac{(\Delta - \Delta^3 - 2d_l + a)}{(\Delta + \Delta^3 + d_l - a)(2\Delta - d_l)} c_I^2 + \frac{3\Delta + \Delta^3 - 3a}{a(3\Delta + \Delta^3 - 2a)} c_I^4 + \frac{a - \Delta - 3\Delta^3}{(\Delta - \Delta^3)(2(\Delta + \Delta^3) - a)} c_I^4 \right] \hfill (C.56)$$

C.5.2.2 $t/u$ channel

Fourier transforming the $t/u$ channel contributions into position space, we find that the leading contribution to the $r'^a - 3\Delta - \Delta^3$ comes with the coefficient,

$$\alpha = \frac{1}{a(a-3\Delta-\Delta^3)} \left[ \frac{h_1 h_3}{\Delta(-a + \Delta + \Delta^3)} - \frac{1}{2h_1 h_2 (-8 + a + \Delta + 3\Delta^3 + 2\epsilon)} \left( -4 + a - \Delta + \Delta^3 + \epsilon)(2(-2 + \Delta + \Delta^3) + \epsilon)(-8 + a + 3\Delta + \Delta^3 + 2\epsilon) \right) \right] \hfill (C.57)$$
Taking the coefficients of $r^{a-3\Delta-\Delta_3}$ from the position space representation of the Green function we get for each exchange ($I = 0, 2$),

$$0 = \sum_{J_0=0,2} c_{I,J_0} \alpha_{J_0} a_{J_0} - \frac{a_I}{a(a - 3\Delta - \Delta_3)} \left[ \frac{(\Delta - \Delta_3 - 2d_I + a)}{(\Delta + \Delta_3 + d_I - a)(2\Delta - d_I)} c_2^I + \frac{3\Delta + \Delta_3 - 3a}{a(3\Delta + \Delta_3 - 2a)} c_3^I \right.
\left. + \frac{a - \Delta - 3\Delta_3}{(\Delta - \Delta_3)(2(\Delta + \Delta_3) - a)} c_4^I \right].$$

(C.58)

The coefficients $\alpha_J$ will contain information about the mixing in the $t$ and $u$ channel as given in (C.57), while $c_{I,J}$ are the mixing coefficients given in (C.21) and the other coefficients are given in (C.47). The coefficients $a_i$ contains the product of the OPE coefficients coming from two different OPE one with $\phi_i \times \phi_j$ and other with $\phi_i \times (\phi^2 \phi_j)$. With this it should be possible to calculate the anomalous dimensions for the operators $\phi_i$ and $\phi^2 \phi_i$ provided we know the anomalous dimensions of the exchanges up to required order in $\epsilon$. 
Appendix D

Details of Chapter 7

D.1 The Lagrangian in terms of [5]

Consider the background field expanded Lagrangian given by

\[ L = a_0 + a_2 \Delta R + b_1 \Delta R^2 + b_2 \Delta R^{ab} \Delta R_{ab} + b_3 \Delta R^{abcd} \Delta R_{abcd} + \sum_{i=1}^{8} \bar{c}_i \Delta K_i \]

\[ + Z^{efabcdmprs} \nabla_e \nabla_f \Delta R_{mprs} \Delta R_{abcd} + \ldots \]  

(D.1)

The Wald functional for any gravity dual following [7] is,

\[ E_{abcd} = \frac{\partial L}{\partial R_{abcd}} - \nabla_e (\frac{\partial L}{\partial \nabla_e R_{abcd}}) + \ldots \],

(D.2)

which for the above Lagrangian takes the form

\[ E_{abcd} = a_2 g^{(ab} g^{cd)} + Y^{abdefg} \Delta R_{efgh} - Z^{efabcdmprs} \nabla_e \nabla_f \Delta R_{mprs} + \ldots \],

(D.3)

where \( Y^{abdefg} \) is the tensor structure that comes from the second order terms in \( \Delta R \) and \( Z^{efabcdmprs} \) comes from the \( \nabla_a R_{bcde} \) terms in the Lagrangian. To connect this to eq.(6.8, 6.9, 6.11) of [5] we need to evaluate \( E_{abcd} \) and \( \delta E_{abcd} \) around AdS space. We split the background metric as \( g_{ab} = g_{(0)ab} + \Delta g_{ab} \). Then \( R_{abcd} \) can be written as

\[ R_{abcd} = -(g_{ac} g_{bd} - g_{ad} g_{bc}) = -(g_{(0)ac} g_{(0)bd} - g_{(0)ad} g_{(0)bc}) - (g_{(0)ab} \Delta g_{cd}) = R^{(0)}_{abcd} - (g_{(0)ab} \Delta g_{cd}), \]

(D.4)

and

\[ \Delta R_{abcd} = R_{abcd} - R^{(0)}_{abcd} = \Delta^0 R_{abcd} + g_{(0)ab} \Delta g_{cd}, \]

(D.5)
where $\Delta^0 R_{abcd} = R_{abcd} - R_{abcd}^{(0)}$ is the expansion around the background with only $g^{(0)ab}$. Using this relation with the fact that

$$Y_{abcdefh} = Y_{0}^{abcdefh} + O(\Delta g),$$

$$Z_{efabcdmprs} = Z_{0}^{efabcdmprs} + O(\Delta g),$$

where $Y_0$ and $Z_0$ denote the quantities calculated with the metric $g^{(0)}$ which is the AdS metric for our purpose. To begin with, we consider an action without the $\nabla R$ terms. Then

$$E_{R}^{abcd} = a_2 g^{(ab,cd)} + Y_{0}^{abcdefh} \Delta^0 R_{efgh} + O(\Delta g),$$

where we have used the tensor structure of $Y$ as

$$Y_{abcdefh} = b_1 g^{ac}g^{bd}g^{eg}g^{fh} + b_2 g^{ac}g^{bg}g^{bf}g^{dh} + b_3 g^{ac}g^{bf}g^{cg}g^{dh},$$

and when evaluated around the AdS background we have at the leading order

$$E_{R}^{abcd} = a_2 g_0^{(ab,cd)}. $$

Comparing with eq (6.8) of [5] we have $a_2 = c_1$. Next we compute

$$\frac{\partial E_{R}^{abcd}}{\partial g^{ef}} = 2c_1 h^{(ab,cd)} + \frac{\partial}{\partial g^{ef}} (Y \Delta R).$$

The last term gives around the AdS background

$$\frac{\partial}{\partial g^{ef}} (Y \Delta R) = \frac{\partial Y}{\partial g} \Delta^0 R + \frac{\partial}{\partial g^{mn}} (Y_{0}^{abcdefh} g_{(0)ef} \Delta g_{gh}).$$

The first term vanishes when evaluated on AdS and thus

$$\frac{\partial}{\partial g^{ef}} (Y \Delta R)|_{AdS} h^{ef} = 2(2b_1 + b_2)h g^{(ab,cd)} + 2((d - 1)b_2 + 4b_3)h^{(ab,cd)}.$$  

Further

$$\frac{\partial E_{R}^{abcd}}{\partial R_{efgh}}|_{AdS} \delta R_{efgh} = Y_{0}^{abcdefh} \delta R_{efgh} = b_1 R_{g}^{(ab,cd)} + b_2 R_{g}^{(ab,cd)} + b_3 R^{abcd},$$

Comparing with eq (6.11) of [5] we get $b_1 = c_4/2$, $b_2 = c_5/2$, $b_3 = c_6/2$. Further, $c_2 = -2 dc_4 - c_5$, $c_3 = 2c_1 - (d - 1)c_5 - 4c_6$. Further,

$$c_1 = 0, c_2 = -c_3, c_3 = 2c_1 - (d - 1)c_5 - 4c_6.$$  

The Lagrangian (7.7) thus can be written as,

$$S = \int d^{d+1}x \sqrt{g}[c_0 + c_1 \Delta R + \frac{c_4}{2} \Delta R^2 + \frac{c_5}{2} \Delta R^{ab} \Delta R_{ab} + \frac{c_6}{2} \Delta R^{abcd} \Delta R_{abcd} + \sum_{i=1}^{8} \tilde{c}_i \Delta K_i + Z \Delta R \nabla \nabla \Delta R + \cdots].$$
D.2 Details of calculation for section (7.3.2)

The Bianchi identity reads

\[ \nabla^a R_{bcde} + \nabla_b R_{acde} + \nabla_c R_{abde} = 0. \]  
(D.15)

Then

\[ \nabla^2 R_{bcde} = \nabla^a \nabla_b R_{acde} - \nabla^a \nabla_c R_{abde}. \]  
(D.16)

Using

\[ \nabla^a \nabla_b R_{acde} = \nabla_b \nabla^a R_{acde} + R^f_{bd} R_{fcede} + R^a_{bd} f R_{cde} + R^a_{bc} f R_{afde} + R^a_{bd} f R_{acdf}, \]  
(D.17)

we have

\[ R^{bcede} \nabla^2 R_{bcde} = 2 R^{bcede} \nabla_b \nabla^a R_{acde} + 2 R^{bcede} R^f_{bd} R_{fcede} + 2 R^{bcde} R^a_{bc} f R_{afde} + 4 R^{bcede} R^a_{bd} f R_{acde}. \]  
(D.18)

Again using the Bianchi Identity,

\[ \nabla^a R_{acde} = \nabla_d R_{ce} - \nabla_e R_{cd}, \]  
(D.19)

we can write, neglecting the total derivatives

\[ R^{bcede} \nabla^2 R_{bcde} = 4 R^{bcede} \nabla_b \nabla^a R_{acde} + 2 R^{bcede} R^f_{bd} R_{fcede} + 2 R^{bcde} R^a_{bc} f R_{afde} + 4 R^{bcede} R^a_{bd} f R_{acde}. \]  
(D.20)

The first term can be written as (neglecting total derivatives),

\[ 4 R^{bcede} \nabla_b \nabla^a R_{acde} = -4 \nabla_b R^{bcede} \nabla^a R_{acde} = -4(\nabla^a R_{acde})^2 - 4 R^{ced} \nabla^e \nabla^d R_{ce} , \]  
(D.21)

and

\[ -4 R^{cd} \nabla^d \nabla^e \nabla^a R_{ce} = (\nabla R)^2 - 4 R^{cd} R^e_{de} f R_{ef} - 4 R^e_{a} R^{d} f R_{c} . \]  
(D.22)

D.3 Holographic stress tensor involving \( \nabla R \) terms

Here we consider an extended analysis of [5] to derive the holographic stress tensor including the \( \nabla \ldots \nabla R \) terms in the action. The most general analysis is deferred for future work although from the following analysis it will be clear that the most general case will also work out in an analogous way. We consider the most general term involving two \( \nabla \)s in the action. Such terms after background field expansion are schematically given by

\[ S = \int d^{d+1}x \sqrt{g} Z \nabla \nabla (\Delta R)^n, \]  
(D.23)
where $Z$ contains all the relevant tensor structures. Note that the Wald functional obtained from such a terms will be of the form

$$E^{abcd}_R = \cdots + Z^{ef} \nabla_e \nabla_f (\Delta R)^{n-1} + \cdots \quad (D.24)$$

For $n > 2$, these terms vanish since when we put the background AdS, $\Delta R^{abcd}$ vanishes in the variation of $E^{abcd}_R$. So the only terms at the two $\nabla$s order relevant for the calculation of the holographic stress tensor are schematically given by $\nabla \Delta R \nabla \Delta R$. These terms in the action are:

$$S_{\nabla R} = \int d^{d+1}x \sqrt{g} Z^{efabcdmnrs} \Delta R_{mnrs} \nabla_e \nabla_f \Delta R^{abcd}, \quad (D.25)$$

where as before $Z^{efabcdmnrs}$ contains all possible tensor structures.

We now focus on the derivation of the holographic stress tensor for the action including (7.54). The Wald functional corresponding to this term is given by

$$E^{abcd}_R = d_1 g^{(ab} g^{cd)} \nabla^2 \Delta R + d_2 \nabla^2 \Delta R^{g^{(ab} g^{cd)}} + d_3 \nabla^2 \Delta R^{abcd}, \quad (D.26)$$

and evaluated on the AdS, $E^{abcd}_R = 0$, while the linearized variation of the wald function is given by

$$\delta E^{abcd}_R = \delta (Z^{efabcdmnrs} \nabla_e \nabla_f \Delta R_{mnrs}), \quad (D.27)$$

where the structure of $Z$ for the contributing terms is given by

$$Z^{efabcdmnrs} = g^{ef} (d_1 g^{ac} g^{bd} g^{mr} g^{ns} + d_2 g^{ac} g^{mr} g^{bn} g^{ds} + d_3 g^{am} g^{bn} g^{cr} g^{ds}). \quad (D.28)$$

All indices are raised or lowered with respect to the background AdS metric $g_{ab}$. Thus combined with the original expressions in [5] for $E^{(1)abcd}_R = E^{abcd}_R + E^{abcd}_\nabla = E^{abcd}_R$ and $\delta E^{(1)abcd}_R$ is given by,

$$\delta E^{(1)abcd}_R = -c_2 h^{(ab} g^{cd)} - c_3 h^{(ab} h^{cd)} + c_4 R g^{(ab} g^{cd)} + c_5 R^{(ab} g^{cd)} + c_6 R^{abcd} + \delta E^{abcd}_\nabla. \quad (D.29)$$

D.3.1 \textbf{d=4}

The coefficients (7.46) for $d = 4$ are given by

$$A = -\frac{c_3}{4} - \frac{3c_5}{4} - 5c_6 + 64d_3, \quad B = -\frac{c_2}{2} - 2c_4 - c_5 - 24d_2 + 8d_3, \quad C = -c_2 - 4c_4 + c_5 + 64d_3, \quad D = -c_3 - 3c_5 + 4c_6 - 128d_3. \quad (D.30)$$
Thus the coefficients $A_1$, $A_2$ take the form

$$
A_1 = -24(c_6 - 16d_3)R^2 + \frac{1}{2}(c_1 - c_3 - 3c_5 - 68c_6 + 1024d_3)(\frac{r^2}{3} - R^2),
$$

$$
A_2 = (\frac{c_1}{2} - c_2 - 4c_4 - 2c_5 - 48d_2 + 16d_3)R^2 + \frac{r^2}{6}(c_1 - c_3 + 15c_5 + 4c_6 + 288d_2 + 160d_3).
$$

We can use the tracelessness condition of $h_{\mu\nu}$ viz. $h^{(d)\mu}_\mu = 0$ to eliminate $A_2$ and thus integrate over $A_1$ to get

$$
d\delta S_{\text{wald}}^B = 8\pi\Omega_2 L R^4 \frac{1}{15}(c_1 + 4c_6 - 64d_3),
$$

where $\Omega_2$ is the volume of the unit $S_2$ and finally using (7.38), we have

$$
\delta T_{tt}^{\text{grav}} = 4\bar{L}[c_1 + 4(c_6 - 16d_3)].
$$

D.3.2 $d=6$

The corresponding coefficients in (7.46) for $d = 6$ after putting $\Delta = d$ are given as

$$
A = \frac{1}{4}(-c_3 - 5c_5 - 52c_6 + 1152d_3),
B = -\frac{c_2}{2} - 3c_4 - \frac{11}{4}c_5 - 60d_2 + 12d_3,
C = -c_2 - 6c_4 + 2c_5 - 120d_2,
D = -c_3 - 5c_5 + 8c_6 - 288d_3.
$$

Putting these in the integral we have

$$
A_1 = \frac{1}{2}(c_1 - c_3 - 5c_5 - 292c_6 + 6912d_3)(\frac{r^2}{5} - R^2) - 120(c_6 - 24d_3)R^2,
$$

$$
A_2 = \frac{1}{2}(c_1 - 2c_2 - 12c_4 - 11c_5 - 240d_2 + 48d_3)R^2 + \frac{1}{2}(c_1 - c_3 + 70c_5 + 8c_6 - 528d_3)\frac{r^2}{5}.
$$

Again by using the tracelessness argument we can set $h = 0$ and integrating and finally using (7.38), we get,

$$
\delta T_{tt}^{\text{grav}} = \frac{35}{2\pi\Omega_4} \lim_{R \to 0} \left( \frac{1}{R^6} \delta S_{\text{wald}}^B \right) = 6\bar{L}^3[c_1 + 8(c_6 - 24d_3)].
$$

D.4 $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle$ in even dimensions

The B-type anomaly coefficients appearing in the expression for the holographic stress tensor in even dimensions are precisely the coefficients of the stress tensor two point functions from the field theory perspective. The 2d and 4d cases were worked out in [14]. We will extend this result to 6d in what follows. Before that we will review the 2d and 4d results.
The starting point of the derivation is the renormalization group equation in [13], [14] which takes on the form in general $d$ dimensions as

$$ (\mu \partial_\mu + 2 \int d^d x g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}}) W = 0. \quad \text{(D.37)} $$

We know that

$$ \int d^d x g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} W = \int d^d x (T_{\mu \nu}) = \int d^d x A_{\text{anomaly}}, \quad \text{(D.38)} $$

which gives us

$$ \mu \partial_\mu W = -2 \int d^d x A_{\text{anomaly}}. \quad \text{(D.39)} $$

We now functionally differentiate the LHS w.r.t $g^{\mu \nu}$ twice to get

$$ \mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = -2 \int d^d x \frac{\delta^2 A_{\text{anomaly}}}{\delta g^{ab} \delta g^{cd}}. \quad \text{(D.40)} $$

From the general conformal properties of the 2 point functions the RHS now takes the form

$$ \mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = \frac{C_T}{4(d-2)^2(d+1)} \Delta^T_{abcd} \mu \partial_\mu \frac{1}{x^{2d-4}}, \quad \text{(D.41)} $$

where the tensor $\Delta^T_{abcd}$ now takes the form

$$ \Delta^T_{abcd} = \frac{1}{2} (S_{ac}S_{bd} + S_{ad}S_{bc}) - \frac{1}{d-1} S_{ab}S_{cd}, \quad \Delta^T_{aaac} = 0, \quad \text{(D.42)} $$

where $S_{ab} = \partial_a \partial_b - \delta_{ab} \partial^2$. In general $x^{-2d+4}$ is singular function. We need to regularize the function in what follows.

### D.4.1 $d=2$

We consider the anomaly in $d = 2$ which is given by $E_2 = \frac{\pi}{4} R$, $I_2 = 0$. The RG equation is given by

$$ \mu \partial_\mu W + \int d^2 x \langle T_i^i \rangle = 0. \quad \text{(D.43)} $$

and the 2 pt function is given by

$$ \mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = \frac{c}{24\pi} \int d^2 x \frac{\delta^2 R}{\delta g^{ab} \delta g^{cd}}. \quad \text{(D.44)} $$

From the second order variation of $R$, $\delta^2 R = h \partial^2 h - h \partial_e \partial_f h^{ef}$ we get,

$$ \frac{\delta^2 R}{\delta g^{ab} \delta g^{cd}} = [(g_{ab} \partial_c \partial_d + g_{cd} \partial_a \partial_b) - g_{ab}g_{cd} \partial^2] \delta^2(x). \quad \text{(D.45)} $$

Converting into the momentum space we can see that

$$ \mu \partial_\mu \langle T_{ab}(p) T_{cd}(0) \rangle = \frac{c}{24\pi} \left[ (g_{ab}p_c p_d + g_{cd}p_a p_b) - g_{ab}g_{cd}p^2 \right], \quad \text{(D.46)} $$
using which we see that $C_T$ and $c$ are proportional to one another.

**D.4.2 $d=4$**

In 4d there are two anomalies given by

$$
E_4 = R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2,
I_4 = E_4 + 2 (R^{ab} R_{ab} - \frac{1}{3} R^2).
$$

(D.47)

The contribution from the $E_4$ term in 4d is given by the integral of

$$
\int d^4 x A^E_{\rho\sigma,\alpha\beta} (x - y, x - z),
$$

(D.48)

where the term $A^E_{\rho\sigma,\alpha\beta} (x - y, x - z)$ is given by

$$
A^E_{\rho\sigma,\alpha\beta} (x - y, x - z) = -(\epsilon_{\sigma\alpha\gamma\kappa} \epsilon_{\rho\beta\delta\lambda} \delta_\kappa \delta_\lambda (\partial_\gamma \delta^d (x - y) \partial_\delta \delta^d (x - z)) + \epsilon_{\rho\sigma\gamma\kappa} \epsilon_{\beta\delta\lambda} \delta_\kappa \delta_\lambda (\partial_\gamma \delta^d (x - y) \partial_\delta \delta^d (x - z))).
$$

(D.49)

To compute the integral we first convert the $\delta^d (x - y)$ into momentum space and carry out the differentiations as

$$
\partial_\delta \delta^d (x - y) \partial_\gamma \delta^d (x - z) = \partial_\gamma (-p_\gamma q_\delta \int e^{i(p+q)x - ipy - iqz} d^d p d^d q).
$$

(D.50)

Acting $\partial_\kappa \partial_\lambda$ on this, we get

$$
\partial_\kappa \partial_\lambda (-p_\gamma q_\delta \int e^{i(p+q)x - ipy - iqz} d^d p d^d q) = p_\gamma q_\delta (p + q)_\lambda (p + q)_\kappa \int e^{i(p+q)x - ipy - iqz} d^d p d^d q.
$$

(D.51)

Thus the first term on the rhs in (D.49) becomes

$$
\epsilon_{\sigma\alpha\gamma\kappa} \epsilon_{\rho\beta\delta\lambda} p_\gamma q_\delta (p + q)_\lambda (p + q)_\kappa \int e^{i(p+q)x} d^d x \int e^{-ipy - iqz} d^d p d^d q,
$$

(D.52)

which becomes after substituting the delta function from the first integral as

$$
\epsilon_{\sigma\alpha\gamma\kappa} \epsilon_{\rho\beta\delta\lambda} \int p_\gamma q_\delta (p + q)_\lambda (p + q)_\kappa \delta^d (p + q) e^{-ipy - iqz} d^d p d^d q.
$$

(D.53)

Thus this integral vanishes on its own. Similarly it can be shown that the second part of the integral also vanishes by itself. Thus there is no contribution from the $E_4$ term to the anomaly. The only contribution to the anomaly comes from the term $R^{ab} R_{ab} - \frac{1}{3} R^2$ term in the Weyl anomaly. Thus

$$
\mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = \frac{c}{16 \pi^2} \int d^4 x \frac{\delta^2}{\delta g^{ab} g^{cd}} [2(R^{mn} R_{mn} - \frac{1}{3} R^2)].
$$

(D.54)

The last term on the rhs gives

$$
\frac{\delta^2 R^2}{\delta g^{ab} \gamma^{cd}} = 2 \frac{\delta R}{\delta g^{ab}} \frac{\delta R}{\delta g^{cd}}.
$$

(D.55)
After linearization of the scalar and functionally differentiating w.r.t $g^{ab}$ we have
\[
\frac{\delta R}{\delta g^{ab}} = (\partial_a \partial_b - g_{ab} \partial^2) \delta^4(x) = S_{ab} \delta^4(x),
\] (D.56)
where we define $S_{ab} = \partial_a \partial_b - g_{ab} \partial^2$. Thus the last term becomes after some integration by parts
\[
\frac{\delta^2 R^2}{\delta g^{ab} \delta g^{cd}} = S_{ab} S_{cd} \delta^4(x) \quad (D.57)
\]
The first term on the rhs becomes after integration by parts as
\[
\frac{\delta R_{mn}}{\delta g^{ab}} \frac{\delta R_{mn}}{\delta g^{cd}} = \frac{1}{2} (S_{ac} S_{bd} + S_{ad} S_{bc}) \delta^4(x).
\] (D.58)
Thus the total contribution from the Weyl anomaly is given by
\[
\mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = -4\beta \Delta^T_{abcd} \delta^4(x),
\] (D.59)
where we define $\Delta^T_{abcd} = \frac{1}{2} (S_{ac} S_{bd} + S_{ad} S_{bc}) - \frac{1}{3} S_{ab} S_{cd}$.
Thus in 4d we have using $\beta = -c/16\pi^2$ from [13]
\[
\mu \partial_\mu \langle T_{ab}(x) T_{cd}(0) \rangle = \frac{c}{4\pi^2} \Delta^T_{abcd} \delta^4(x).
\] (D.60)
Comparing with (D.41) we have
\[
\frac{C_T}{4(d-2)^2 d(d+1)} \mu \partial_\mu \frac{1}{x^4} = \frac{c}{4\pi^2} \delta^4(x).
\] (D.61)
In 4d the regularized $1/x^4$ can be expressed as
\[
R \frac{1}{x^4} = -\frac{1}{4} \partial^2 \frac{1}{x^2} (\log \mu^2 x^2) \Rightarrow \mu \partial_\mu R \frac{1}{x^4} = 2\pi^2 \delta^4(x).
\] (D.62)
Putting this in (D.61) we have
\[
c = \frac{\pi^4}{40} C_T.
\] (D.63)

**D.4.3 d=6**

In 6d it is rather easy to see why only the $B_3$ coefficient gets picked up by the 2 pt functions. If we look at the structures of the anomalies then only $I_3$ has a structure of the form
\[
I_3 \sim C^{abcd} \partial^2 C_{abcd}.
\] (D.64)
This makes $I_3$ to start at the order $O(h^2)$ and contributes in the 2 pt function. While all the other anomalies start at $O(h^3)$ and thus do not contribute.
In 6d the only contribution to the two point function comes from the term \( I_3 \sim C_{abcd} \partial^2 C_{abcd} \), since the other anomalies start at \( O(h^3) \). Thus from (D.41) we have

\[
\mu \partial_\mu \langle T_{ab}(x)T_{cd}(0) \rangle = 6B_3 \Delta^T_{\text{abcd}} \partial^2 \delta^6(x) = \Delta^T_{\text{abcd}} \frac{C_T}{2^7 \times 3 \times 7} \mu \partial_\mu \frac{1}{x^8}. \tag{D.65}
\]

In 6d we regularize as

\[
\mathcal{R} \frac{1}{x^8} = -\frac{1}{96} \partial^4 \frac{1}{x^4} (\log \mu^2 x^2) \Rightarrow \mu \partial_\mu \mathcal{R} \frac{1}{x^8} = -\frac{1}{48} \partial^4 \frac{1}{x^4}. \tag{D.66}
\]

The term on the RHS for 6d can be reduced to \( \partial^4 \frac{1}{x^4} = -\frac{\pi^2}{12} \partial^2 \delta^6(x) \) and hence the RHS becomes

\[
\mu \partial_\mu \mathcal{R} \frac{1}{x^8} = -\frac{1}{48} \partial^4 \frac{1}{x^4} = \frac{\pi^3}{12} \partial^2 \delta^6(x). \tag{D.67}
\]

Thus in 6d we have

\[
\mu \partial_\mu \langle T_{ab}(x)T_{cd}(0) \rangle = 6B_3 \Delta^T_{\text{abcd}} \partial^2 \delta^6(x) = \frac{\pi^3}{7 \times 3 \times 2^9} \Delta^T_{\text{abcd}} C_T \partial^2 \delta^6(x). \tag{D.68}
\]

Hence we have

\[
B_3 = \frac{\pi^3}{64 \cdot 7!} \frac{5}{3} C_T. \tag{D.69}
\]

### D.5 \( \eta/s \) for general \( R^2 \) theories

We will calculate the ratio of the shear viscosity to the entropy density for four derivative theory of gravity in \( d = 4 \) where \( d \) is the boundary dimension. We want to express the ratio in terms of the field theory variables as \( t_2 \) etc. This analysis can be extended for general higher derivative theories of gravity in arbitrary dimensions. To proceed we will follow the analysis of [31] where the horizon is first constructed perturbatively and then the pole method was used to extract the shear viscosity. We first consider the metric as

\[
ds^2 = \frac{\tilde{L}^2}{\tilde{f}(z)} \frac{dz^2}{(1-z)^2} + \frac{r_0^2}{L^2(1-z)} \left[ -\frac{f(z)}{f_\infty} dt^2 + (dx_1 + \phi(t)dx_2)^2 + dx_2^2 + dx_3^2 \right]. \tag{D.70}
\]

where \( \phi(t) = e^{-i\omega t} \) is the fluctuation and

\[
f(z) = 2z + f_2 z^2 + f_3 z^3 + \ldots. \tag{D.71}
\]

We consider the general \( R^2 \) action given by

\[
S = \int d^5x \sqrt{g} \left[ R + \frac{12}{L^2} + \frac{L^4}{2} (\lambda_1 R^{abcd} R_{abcd} + \lambda_2 R^{ab} R_{ab} + \lambda_3 R^2) \right]. \tag{D.72}
\]

To obtain the coefficients \( f_2, f_3 \) we plug in (D.71) into the equations of motion for the action (D.72) and solve perturbatively near the horizon. The solution for \( f_\infty \) taking \( L = \tilde{L} \sqrt{f_\infty} \) given
by,
\[ 1 - f_\infty + \frac{1}{3}(\lambda_1 + 2\lambda_2 + 10\lambda_3)f_\infty^2 = 0. \]  
(D.73)

The expression for \( c_1 \) is given by
\[ c_1 = \frac{1}{16\pi} [1 - 2f_\infty(\lambda_1 + 2\lambda_2 + 10\lambda_3)]. \]  
(D.74)

We also express \( c_6 = \frac{\lambda_1}{16\pi} f_\infty \). The shear viscosity and entropy density in terms of these couplings are given by
\[
\eta = C_1 [8\lambda_1^2 + \lambda_2 + 4\lambda_3 - 20\lambda_2\lambda_3 - 64\lambda_3^2 + 12\lambda_1(\lambda_2 + 2\lambda_3) + \sqrt{\mathcal{F}}], \\
s = C_2 [16\lambda_1^2 + \lambda_2 + 4\lambda_3 + 20\lambda_2\lambda_3 + 48\lambda_1\lambda_3 - 20\lambda_2\lambda_3 - 64\lambda_3^2 + \sqrt{\mathcal{F}}],
\]  
(D.75)

where the normalizations are \( C_2 = \frac{2\pi r_0 f_\infty^3}{L} \) and \( C_1 = \frac{r_0 f_\infty^3}{2L} \) and we have set \( L = 1 \).

\[
\mathcal{F} = (2\lambda_1 + \lambda_2 + 2\lambda_3)(2\lambda_1 + \lambda_2 + 2\lambda_3)(1 - 12\lambda_1 - 16\lambda_2 - 52\lambda_3)^2 \\
- 16(\lambda_1 + \lambda_2 + 2\lambda_3)(22\lambda_1^2 - \lambda_2(1 - 12\lambda_2) - 2\lambda_3 + 62\lambda_2\lambda_3 + 70\lambda_3^2 - 2\lambda_1(1 - 19\lambda_2 - 58\lambda_3)).
\]  
(D.76)

The corresponding expression for \( t_2 \) is \( d = 4 \) is given by
\[
t_2 = \frac{24c_6}{c_1 + 4c_6} = \frac{24\lambda_1 f_\infty}{1 + 2(\lambda_1 - 2\lambda_2 - 10\lambda_3)f_\infty}. \]  
(D.77)

Note that in the limit \( \lambda_1, \lambda_2, \lambda_3 \to 0 \) we retrieve the result
\[
\frac{\eta}{s} = \frac{1}{4\pi}. \]  
(D.78)

Note also that it is possible to make \( \eta/s \) arbitrarily small by tuning the values of \( \lambda s' \). For example for \( \lambda_1 = 0.31517, \lambda_2 = \lambda_3 = -1 \), we have
\[
\frac{\eta}{s} = \frac{1.1 \times 10^{-5}}{4\pi}. \]  
(D.79)

Here the constraints arising from \( \langle \epsilon \rangle > 0 \) are satisfied.