The Hadronic Light-by-Light Scattering Contribution to the Muon $g-2$

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Contents

Nyffeler, arXiv:0901.1172 [hep-ph]
Jegerlehner + Nyffeler, in preparation
Note: some numbers are still preliminary !

Strong Frontier 2009
Bangalore, India, 12-17 January 2009

1. Introduction: Basics of the anomalous magnetic moment

For a spin 1/2 particle:

$$\vec{\mu} = g\frac{e}{2m}\vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1 + a), \qquad a = \frac{1}{2}(g - 2) : \text{anomalous magnetic moment}$$

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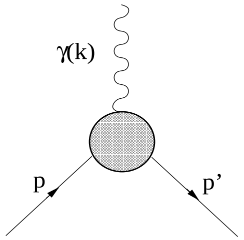
In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{u}(p') \left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu} k_\nu}{2m} \underbrace{F_2(k^2)}_{\text{Pauli}} \right] u(p)$$

$$F_1(0) = 1 \quad \text{und} \quad F_2(0) = a$$

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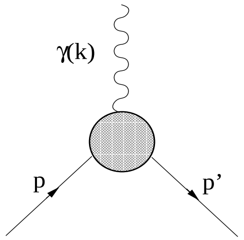

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a_e: Most precise determination of $\alpha = e^2/4\pi$

$$a_e^{\text{exp.}} = (11\,596\,521\,80.73 \pm 0.28) \times 10^{-12} \quad [0.24\text{ppb}] \qquad \text{[Hanneke et al. '08]}$$

a_μ: $a_\mu^{\text{exp}} = (11\,659\,2080 \pm 63) \times 10^{-11}$ [0.5ppm] [g-2 experiment Brookhaven, '06]

All sectors of Standard Model (SM) contribute: sensitive to electroweak and hadronic corrections and to possible contributions from New Physics:

$a_l \sim \left(\frac{m_l}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 40000$ more sensitive than a_e [exp. precision $\longrightarrow$ "only" factor 19]

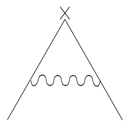
2. Muon $g-2$ in the SM

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$

QED contributions (from leptons) and electroweak contributions (EW) under control

QED (up to 5 loops !):

$$a_\mu^{\text{QED}} = (116\,584\,718.101 \pm 0.148) \times 10^{-11}$$



$+ \ldots$

[Schwinger '48; . . . ; Kinoshita et al.; Laporta + Remiddi '96; Czarnecki + Marciano 2000; . . . ; Kinoshita et al. '07, '08 **!**]

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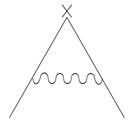

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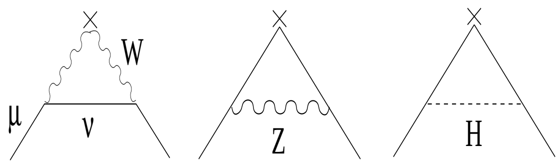
Electroweak contributions (up to 2 loops):

$$a_\mu^{\text{EW},(1)} = (194.82 \pm 0.02) \times 10^{-11}$$



$$a_\mu^{\text{EW},(2)} = (-42.08 \pm 1.80) \times 10^{-11},$$

large since $\sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$

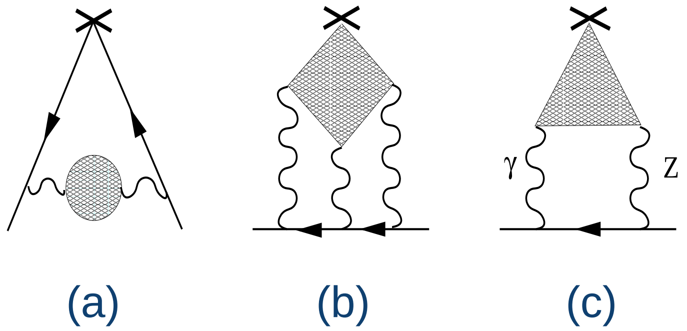
$$a_\mu^{\text{EW}} = (153.2 \pm 1.8) \times 10^{-11}$$

[Brodsky + Sullivan '67; . . . ; Knecht et al. '02; Czarnecki, Marciano, Vainshtein '02]

Hadronic contributions to a_μ

- QCD: quarks bound by strong gluonic interactions into hadronic states
- In particular for the light quarks $u, d, s \longrightarrow$ cannot use perturbation theory !
- Largest source of error in a_μ

Different types of contributions:



Light quark loop not well defined
$\longrightarrow$ Hadronic "blob"

(a) Hadronic vacuum polarization $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3)$

(b) Hadronic light-by-light scattering $\mathcal{O}(\alpha^3)$

(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

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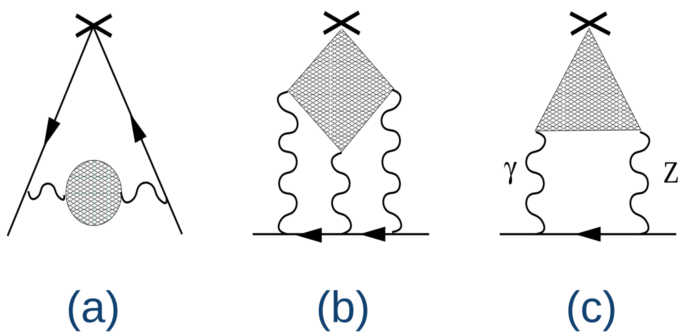

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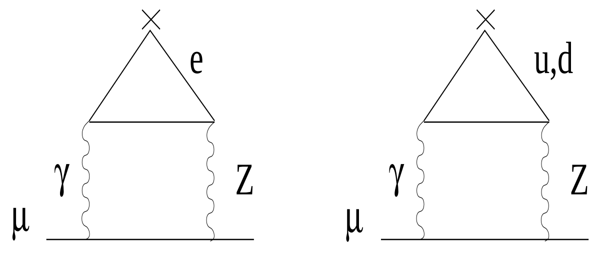
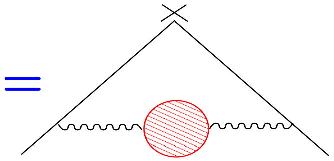
2-Loop EW
Small hadronic uncertainty $\sim 1 \times 10^{-11}$
from triangle diagrams
Anomaly cancellation within each generation !
Cannot separate leptons and quarks !

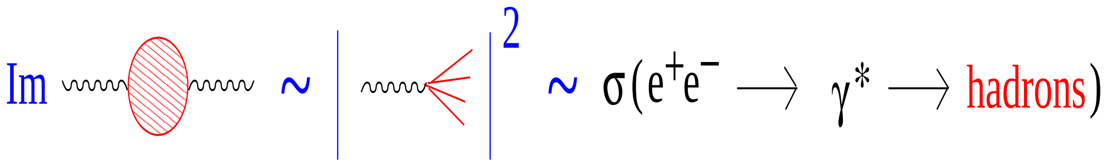


Hadronic vacuum polarization $\mathcal{O}(\alpha^2)$

At lowest order: can use optical theorem (unitarity)

$a_\mu^{\text{had. v.p.}} =$





Starting from spectral representation for 2-point function:

$$a_\mu^{\text{had. v.p.}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

$K(s)$ slowly varying and positive; low-energy data very important due to factor $1/s$
[Bouchiat + Michel ’61; Durand ’62; Brodsky + de Rafael ’68; Gourdin + de Rafael ’69]

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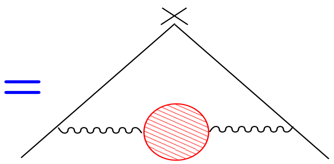
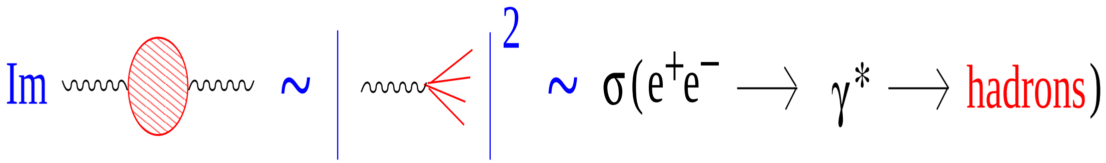
$a_\mu^{\text{had. v.p.}} =$ 



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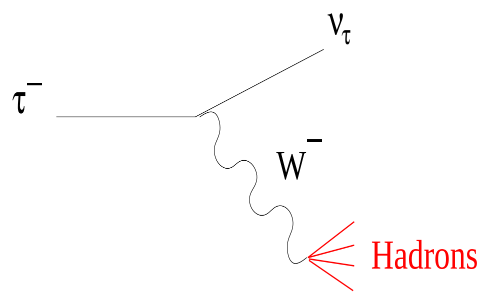
- Hadronic τ-decays e.g. $\tau^- \to \nu_\tau \pi^- \pi^0$
Problem: Corrections due to violation of isospin: $m_u \neq m_d$, electromagnetic radiative corrections
[Alemany et al. ’98; Cirigliano et al. ’01, ’02; Davier et al. ’03]



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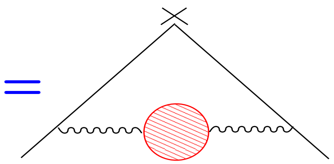
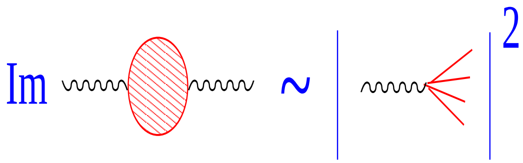
$a_\mu^{\text{had. v.p.}} =$ 



$\sim \sigma(e^+e^- \longrightarrow \gamma^* \longrightarrow \text{hadrons})$

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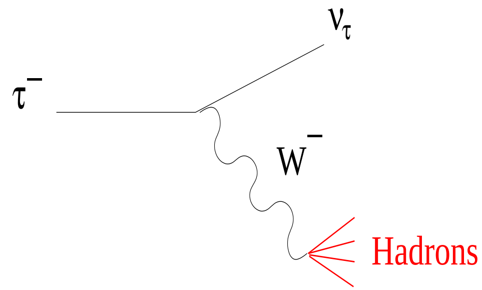
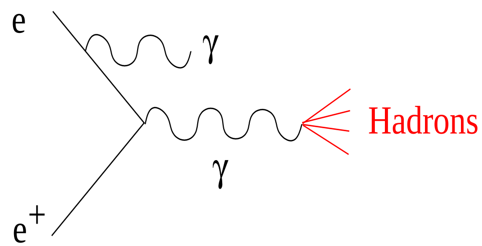


- “Radiative return” at colliders with fixed center of mass energy (DAΦNE, B-Factories, LEP, . . .)
[Binner, Kühn, Melnikov ’99; Czyż et al. ’00-’03]



Values for $a_\mu^{\text{had v.p.}}$

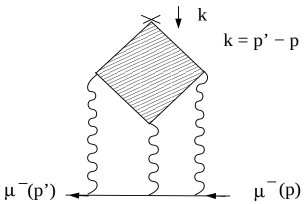
Selection of recent evaluations:

Authors	Contribution to $a_\mu^{\text{had v.p.}} \times 10^{11}$
Davier et al. '03 (e^+e^-)	$6963 \pm 62_{\text{exp}} \pm 36_{\text{rad}}$ $[\pm 72]$
Davier et al. '03 ($e^+e^- + \tau$)	$7110 \pm 50_{\text{exp}} \pm 8_{\text{rad}} \pm 28_{\text{SU(2)}}$ $[\pm 58]$
de Troconiz, Yndurain '05 (e^+e^-)	6935 ± 59
de Troconiz, Yndurain '05 ($e^+e^- + \tau$)	7018 ± 58
Davier et al. '06 (e^+e^-)	6909 ± 44
Hagiwara et al. '07 (e^+e^-, inclusive)	6894 ± 46
Jegerlehner '08 (e^+e^-)	6903.0 ± 52.6

- The last three entries include recent data from SND, CMD-2, BaBar
- Jegerlehner '08 also includes latest data from BaBar and KLOE
- Systematic discrepancy (2-3 σ) of spectral functions obtained from e^+e^- and from τ-data, in particular above the peak of the ρ-meson
- This could be due to an additional source of isospin violation
  Davier '03; Ghozzi + Jegerlehner '04; Benayoun '08
- Because of this uncertainty, τ-data have not been used anymore in recent evaluations of $a_\mu^{\text{had v.p.}}$

Hadronic light-by-light scattering in muon $g-2$

Classification of contributions (de Rafael '94)



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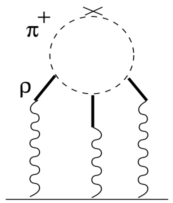

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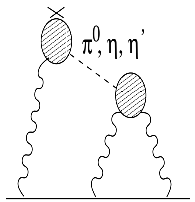
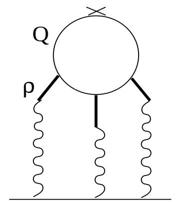


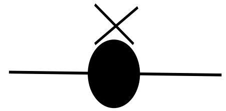
+ --- + Exchange of other resonances $(f_0, a_1, \ldots)$ +



+ ---

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Last term corresponds to local "counterterm" $\sim \bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ in EFT approach to LbyL scattering. Relevant scale in LbyL $\sim$ 500 MeV-2 GeV $\longrightarrow$ need hadronic resonance model $\longrightarrow$ Strong Frontier of QCD !



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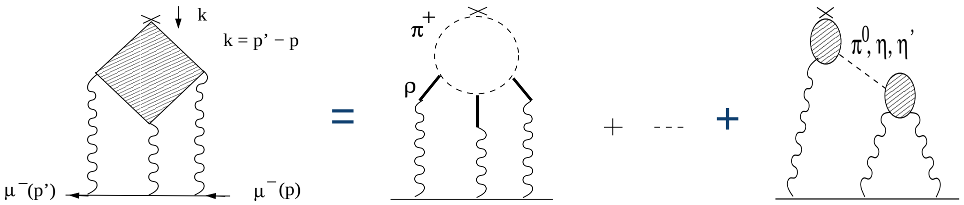

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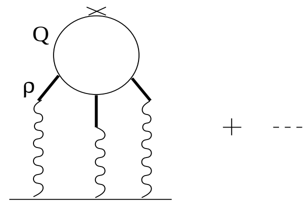

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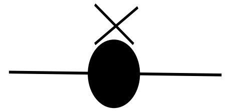


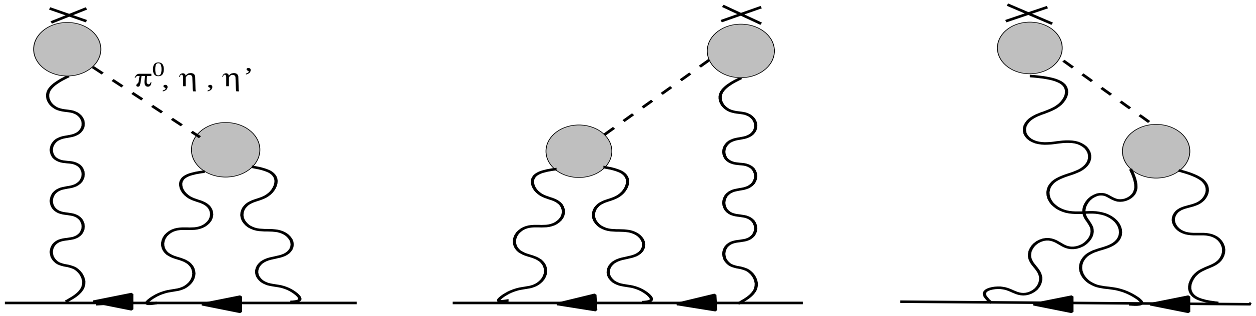
Contribution to $a_\mu \times 10^{11}$

HKS:	+90 (15)	-5 (8)	+83 (6)	+1.74 $[a_1]$	+10 (11)
BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	0
		ud.: -45			ud.: +60

HKS = Hayakawa, Kinoshita, Sanda, BPP = Bijnens, Pallante, Prades, KN = Knecht, Nyffeler
MV = Melnikov, Vainshtein, 2007 = Bijnens, Prades; Miller, de Rafael, Roberts
PdRV = Prades, de Rafael, Vainshtein '09 (new combination of results; no dressed quark loop !)
ud. = undressed, i.e. point vertices without form factors

3. Pion-exchange contribution: a new short-distance constraint

Pseudoscalar-exchange contribution to a_μ:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta'$

Consider π^0. Definition of form factor:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2\cdot y)}\, \langle 0|T\{j_\mu(x) j_\nu(y) P^3(0)\}|0\rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \frac{i\langle\overline{\psi}\psi\rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$$

Up to small mixing effects with the states η and η'

j_μ = light quark part of the electromagnetic current:

$j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$, $\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $\hat{Q}$ = diag$(2,-1,-1)/3$: charge matrix

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2$, $\langle\overline{\psi}\psi\rangle$ = single flavor quark condensate

Bose symmetry: $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

Pion-exchange contribution

Projection onto the muon $g-2$ leads to (Knecht + Nyffeler '01; Jegerlehner '07, '08):

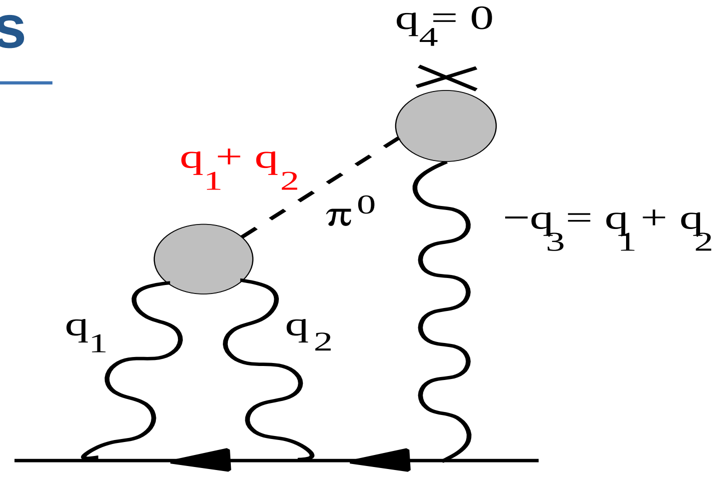
$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} &= -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\times \left[\frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^0*\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} T_1(q_1, q_2; p) \right. \\
&\left. + \frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)\ \mathcal{F}_{\pi^0*\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} T_2(q_1, q_2; p) \right]
\end{aligned}
$$

$$
\begin{aligned}
T_1(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 \\
&- \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 \\
&+ \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2 \\
T_2(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2 \\
&+ \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2 \\
&+ \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon)

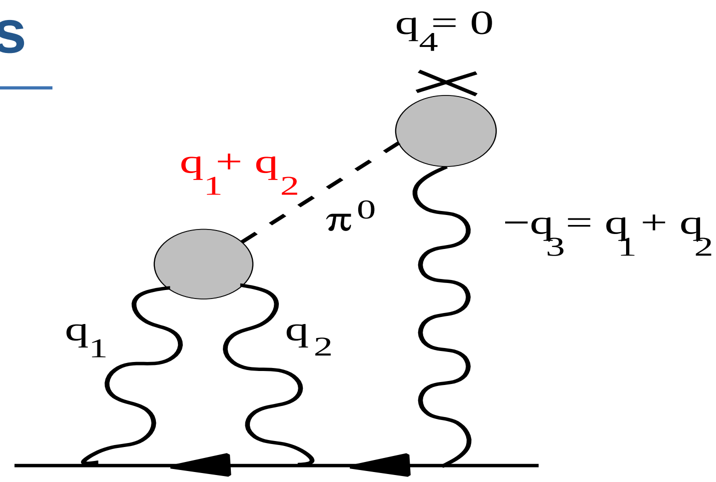
Off-shell versus on-shell form factors

- Jegerlehner '07, '08 pointed out that one should use off-shell form factors for the evaluation of the pion-exchange contribution (seems to have been overlooked in the recent literature). For instance, from the second diagram contributing to the term with T_1 (with appropriately relabeled momenta):



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

Off-shell versus on-shell form factors



- Jegerlehner '07, '08 pointed out that one should use off-shell form factors for the evaluation of the pion-exchange contribution (seems to have been overlooked in the recent literature). For instance, from the second diagram contributing to the term with T_1 (with appropriately relabeled momenta):

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- On the other hand, Knecht + Nyffeler '01, Bijnens + Persson '01 and maybe other references used on-shell form factors, which leads to

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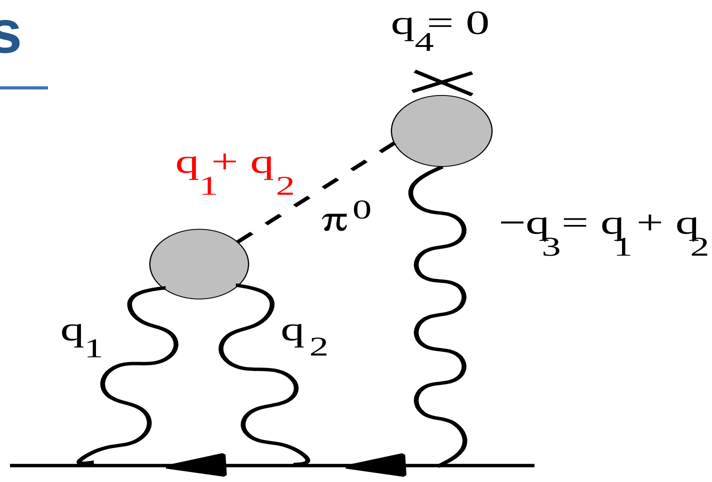

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- Melnikov + Vainshtein '03 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term

- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution !

Off-shell versus on-shell form factors (cont.)

- In general, any evaluation e.g. using some resonance Lagrangian will lead to off-shell form factors at the vertices and will therefore not yield the pion-pole contribution only.

- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell (or nearly on-shell).

  If one is (far) off the mass shell of the exchanged particle, it is not possible to separate different contributions to the $g-2$, unless one uses some particular model where for instance elementary pions can propagate.

  Although the contribution in a particular channel will then be model-dependent, the sum of all off-shell contributions in all channels will lead, at least in principle, to a model-independent result.

- Following Jegerlehner, we will later use off-shell form factors

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

  We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. '89), which also fulfills all the relevant QCD short-distance constraints.

Experimental constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

1. Any hadronic model of the form factor has to reproduce the decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}\left[1 + \mathcal{O}\left(m_q\right)\right]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small)

This leads to the constraint

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2. Information on the form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Experimental data (CELLO '90, CLEO '98) fairly well confirm the Brodsky-Lepage behavior:

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor

QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht + Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

1. If the space-time arguments of all three currents approach each other one obtains (up to corrections $\mathcal{O}(\alpha_s)$)

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

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2. When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other (OPE leads to Green’s function $\langle AP \rangle$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, (\lambda q_1)^2, (q_2 - \lambda q_1)^2) = \frac{2F_0}{3} \frac{1}{\lambda^2} \frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

Higher twist corrections have been worked out in Shuryak + Vainshtein ’82, Novikov et al. ’84 (in chiral limit):

$$\lim_{\lambda\to\infty} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, (\lambda q_1)^2, (\lambda q_1)^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0,0,0)} = -\frac{8}{3}\pi^2 F_0^2 \left\{ \frac{1}{\lambda^2 q_1^2} + \frac{8}{9}\frac{\delta^2}{\lambda^4 q_1^4} + \mathcal{O}\left(\frac{1}{\lambda^6}\right) \right\}$$

δ^2 parametrizes the relevant higher-twist matrix element

The sum-rule estimate in Novikov et al. ’84 yielded $\delta^2 = (0.2 \pm 0.02)$ GeV^2

A new short-distance constraint at the external vertex in $a_\mu^{\mathrm{LbyL};\pi^0}$

3. When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP \rangle$ one obtains (Knecht + Nyffeler, EPJC '01):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

where the vector-tensor two-point function Π_{VT} is defined by

$$\delta^{ab}(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\,\psi)(0)\}|0\rangle$$

with $\sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$. Conservation of the vector current and parity invariance then give

$$(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma} - p_\sigma\eta_{\mu\rho})\,\Pi_{\mathrm{VT}}(p^2)$$

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At the external vertex in light-by-light scattering the following limit is relevant (soft photon $q_2 \longrightarrow 0$)

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3}\frac{F_0}{\langle \overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Note that there is no fall-off in this limit, unless $\Pi_{\mathrm{VT}}(0)$ vanishes !

A constituent quark model for the form factor would lead to a $1/q_1^2$ fall-off instead

New short-distance constraint at the external vertex (cont.)

Ioffe + Smilga '84 defined the quark condensate magnetic susceptibility χ of QCD in the presence of a constant external electromagnetic field

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\, e_q\, \chi\, \langle\overline{\psi}\psi\rangle_0\, F_{\mu\nu}, \qquad e_u = 2/3,\; e_d = -1/3$$

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$$\Pi_{\mathrm{VT}}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\,\chi$$

with our conventions for Π_{VT}, see also Mateu + Portoles ’07

Therefore we get at the external vertex in $a_\mu^{\mathrm{LbyL};\pi^0}$

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- Unfortunately there is no agreement in the literature what the actual value of χ should be.
- In comparing different results one has to keep in mind that χ actually depends on the renormalization scale μ.
- A quantitative comparison of various results for $\chi(\mu)$ is difficult, since one cannot use perturbative QCD down to scales of $\mu \sim 0.5$ GeV at which some estimates are given.
- Finally, it is not clear what would be the “correct” scale μ in the context of hadronic light-by-light scattering in the muon $g-2$!

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe + Smilga '84	QCD sum rules	$\chi(\mu = 0.5$ GeV$) = -(8.16^{+2.95}_{-1.91})$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Balitsky + Yung '83	LMD	$\chi = -2/M_V^2 = -3.3$	[4]
Belyaev + Kogan '84	QCD sum rules	$\chi(0.5$ GeV$) = -(5.7 \pm 0.6)$	[5]
Balitsky et al. '85	QCD sum rules	$\chi(1$ GeV$) = -(4.4 \pm 0.4)$	[5]
Ball et al. '03	QCD sum rules	$\chi(1$ GeV$) = -(3.15 \pm 0.30)$	[5]

[1]: QCD sum rule evalation of nucleon magnetic moments
[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?
[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation
[4]: The leading short-distance behavior of Π_{VT} is given by (see Craigie + Stern '81)

$$\lim_{\lambda\to\infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2}\frac{\langle\overline{\psi}\psi\rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

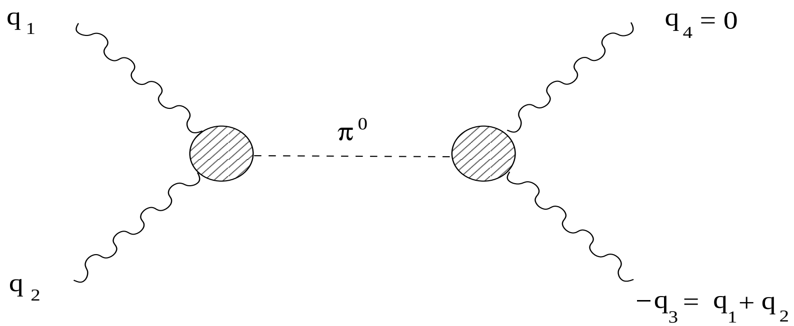
Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky + Yung '83, Belyaev + Kogan '84, Knecht + Nyffeler, EPJC '01)

$$\Pi_{VT}^{LMD}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \quad \chi^{LMD} = -\frac{2}{M_V^2} = -3.3 \text{ GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.
[5]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of Π_{VT}. Note that the last value is very close to original LMD estimate !

The short-distance constraint by Melnikov + Vainshtein

Melnikov + Vainshtein '03 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}}\, V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

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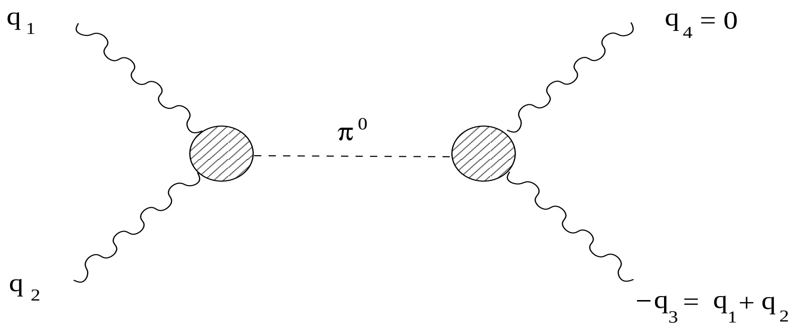

$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

In this way they obtain as intermediate result for the LbyL scattering amplitude:

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = field strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$.

Except in $\tilde{f}_{\rho\sigma}$ the limit $q_4 \to 0$ is understood in $f_3^{\sigma\rho}$ and in the pion propagator.

- From the expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is obvious that Melnikov + Vainshtein only consider the pion-pole contribution !
- Note the absence of a second form factor at the external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by a constant (WZW) form factor $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0)$!
- Note the overall $1/q_3^2$ behavior for large q_3 (apart from $f_3^{\sigma\rho}$)

4. New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic ansatz for Green's function in large-N_C QCD

- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

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Knecht + Nyffeler, EPJC '01, wrote down such minimal hadronic ansätze for on-shell $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ and off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$

- With one multiplet of vector resonances (LMD) or two multiplets ρ and ρ' (LMD+V).
- Both fulfill all OPE short-distance constraints, but the LMD ansatz does not reproduce the Brodsky-Lepage fall-off.

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Off-shell LMD+V form factor:

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2+q_2^2+q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)(q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1\, (q_1^2+q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\, (q_1^2+q_2^2)\, q_3^2 + h_4\, q_3^4 + h_5\, (q_1^2+q_2^2) + h_6\, q_3^2 + h_7$$

where $q_3^2 = (q_1+q_2)^2$

In the spirit of Resonance Chiral Theory (Ecker et al. '89) a Lagrangian with two multiplets of vector resonances was proposed by Mateu + Portoles '07 which reproduces the LMD+V ansatz and the corresponding short-distance constraints.

Fixing the LMD+V model parameters

- h_7: Determined through the normalization with decay width $\pi^0 \to \gamma\gamma$
  $\Rightarrow h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
- h_1: In order to reproduce the Brodsky-Lepage behavior
  $\Rightarrow h_1 = 0$ GeV^2
- h_5: Knecht + Nyffeler, EPJC '01 have fitted the on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$ to the CLEO data
  $\Rightarrow h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$
- h_2: Melnikov + Vainshtein '03 pointed out that h_2 can be obtained from higher-twist corrections in OPE
  $\Rightarrow h_2 = -4\,(M_{V_1}^2 + M_{V_2}^2) + (16/9)\,\delta^2 \simeq -10.63$ GeV^2
  where we used $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV and $\delta^2 = 0.2$ GeV^2

Fixing the LMD+V model parameters (cont.)

- h_3, h_4: Vector-tensor two-point function in LMD+V framework (Knecht + Nyffeler, EPJC '01):

$$\Pi_{\mathrm{VT}}^{\mathrm{LMD+V}}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{\mathrm{VT}}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{\mathrm{VT}} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

OPE constraint for form factor leads to relation

$$h_1 + h_3 + h_4 = 2c_{\mathrm{VT}} = M_{V_1}^2 M_{V_2}^2 \chi$$

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$\chi^{\rm LMD} = -2/M_V^2 = -3.3$ GeV^{-2} is close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} obtained by Ball et al. '03 using QCD sum rules with several vector resonances ρ, ρ', ρ''.
Assuming that the LMD/LMD+V framework is self-consistent, we will take

$$\chi = (-3.3 \pm 1.1)\ \text{GeV}^{-2} \ \Rightarrow \ h_3 + h_4 = -4.3 \pm 1.4\ \text{GeV}^2 \qquad (*)$$

with a typical large-N_C uncertainty of 30% for χ
Will vary h_3 in the range ± 10 GeV^2 and determine h_4 from the relation $(*)$ and vice versa

Fixing the LMD+V model parameters (cont.)

- h_3, h_4: Vector-tensor two-point function in LMD+V framework (Knecht + Nyffeler, EPJC '01):

$$\Pi_{\mathrm{VT}}^{\mathrm{LMD+V}}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{\mathrm{VT}}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{\mathrm{VT}} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

OPE constraint for form factor leads to relation

$$h_1 + h_3 + h_4 = 2c_{\mathrm{VT}} = M_{V_1}^2 M_{V_2}^2 \chi$$

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- Short-distance constraint by Melnikov + Vainshtein '03 (M+V '03). Taking first $q_1^2 \sim q_2^2 \gg q_3^2$ and then q_3^2 large, one obtains at external vertex:

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}^{\mathrm{LMD+V}}(q_3^2, q_3^2, 0) \stackrel{q_3^2\to\infty}{\longrightarrow} \frac{h_1 + h_3 + h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{\mathrm{VT}}}{M_{V_1}^2 M_{V_2}^2} = \chi$$

With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with M+V '03 !

Fixing the LMD+V model parameters (cont.)

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With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with M+V '03 !
With constant (WZW) form factor at external vertex we would get (in chiral limit):

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^0\gamma\gamma}^{\mathrm{LMD+V}}(0,0,0) = \frac{h_7}{M_{V_1}^4 M_{V_2}^4} = -\frac{N_C}{4\pi^2 F_\pi^2} \simeq -8.9 \text{ GeV}^{-2}$$

With $\chi = -N_C/(4\pi^2 F_\pi^2)$ from Vainshtein '03, we would precisely satisfy the short-distance constraint from M+V '03. Problem: M+V '03 only consider pion-pole contribution !

Fixing the LMD+V model parameters (cont.)

- h_6:
  - Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
  - Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.

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  - In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht + Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11 \; \frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4} \frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2} \left(1 + \frac{M_{V_1}^2}{M_{V_2}^2} \right) = -1.36 \; \frac{10^{-4}}{F_\pi^2}$$

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$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26 \, \frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4} h_6$$

Note, however, that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" in size compared to A_{V,p^2}^{LMD}. It has about the same size as the absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

  - Assuming again that the LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = 5 \pm 5$ GeV^4

New estimate for pion-exchange contribution

$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
$h_4 = -10$ GeV^2	65.3	70.6	76.4
$h_4 = 0$ GeV^2	67.2	72.8	78.8
$h_4 = 10$ GeV^2	69.2	75.0	81.2

$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV$^4 - h_3 m_\pi^2$
When varying h_3 (upper half of the table), the parameter h_4 is fixed by the constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$. In the lower half the procedure is reversed.

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- Varying χ by ± 1.1 GeV^{-2} changes the result by $\pm 2.1 \times 10^{-11}$ at most.
- Uncertainty in h_6 affects result by $\pm 6.4 \times 10^{-11}$. If we use $h_6 = (0 \pm 10)$ GeV^4, the result would vary by $\pm 12 \times 10^{-11}$!
- The variation with h_3 (with h_4 determined from the constraint or vice versa) is much smaller, at most $\pm 2.5 \times 10^{-11}$.
- The variation of h_5 by ± 0.26 GeV^4 only leads to changes of $\pm 0.6 \times 10^{-11}$.
- Within scanned region:
  - Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]
  - Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

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$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor

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Take average of results obtained with $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as central value:

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

Added errors from χ, h_3 (or h_4) and h_6 linearly (don't follow Gaussian distribution !)

Update for η and η' using VMD form factor

- Short-distance analysis of LMD+V form factor in Knecht + Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
- Simplified approach (as done in many other papers) using VMD form factors

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

normalized to experimental decay width $\Gamma(\mathrm{PS} \to \gamma\gamma)$

$\Gamma(\eta \to \gamma\gamma) = 0.510 \pm 0.026$ keV $\Rightarrow F_{\eta,\mathrm{eff}} = 93.0$ MeV ($m_\eta = 547.853$ MeV)

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  CLEO fitted VMD ansatz for $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2, 0)$ with adjustable parameter Λ_{PS} instead of $M_V \Rightarrow \Lambda_\eta = 774 \pm 29$ MeV, $\Lambda_{\eta'} = 859 \pm 28$ MeV. We take these masses for our evaluation.

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- With VMD form factors at both vertices (i.e. not taking pole-approximation as in Melnikov+Vainshtein '03) we get $a_\mu^{{\rm LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{{\rm LbyL};\eta'} = 12.5 \times 10^{-11}$
- Our final estimate for the sum of all pseudoscalars:

$$a_\mu^{\rm LbyL;PS} = (99 \pm 16) \times 10^{-11}$$

where we have assumed a 16% error, as inferred for pion-exchange contribution.
A new detailed analysis for η, η' is needed along the lines of LMD+V.

Recent results for the pseudoscalar-exchange contribution

Model for $\mathcal{F}_{P^*\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
Point coupling	$+\infty$	$+\infty$
ENJL (modified) [BPP]	59(9)	85(13)
VMD / HLS [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	–
LMD+V (off-shell)	72(12)	99(16)

BPP = Bijnens, Pallante, Prades '95, '96, '02 (ENJL = Extended Nambu-Jona-Lasinio model)

HK(S) = Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02 (HLS = Hidden Local Symmetry model)

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- Our result is not too far from value given by M+V '03, but this is pure coincidence ! We use off-shell form factors at both vertices, they use on-shell form factors, i.e. a constant factor at the external vertex !

  Note: Following M+V '03 and using $h_2 = -10$ GeV^2 we actually obtain 79.8×10^{-11} for the pion-pole contribution, close to the value 79.6×10^{-11} given in Bijnens + Prades '07 and 79.7×10^{-11} in D+B '08
- In the nonlocal chiral quark model (D+B '08) there is a strong damping for off-shell pions, therefore the result is always smaller than the pion-pole contribution. In the LMD+V model the final result can be bigger or smaller, depending on the model parameters. The error given by D+B '08 probably underestimates intrinsic model uncertainties !

Recent evaluations of hadronic light-by-light scattering

Some recent results for the various contributions to $a_\mu^{\mathrm{LbyL;had}} \times 10^{11}$

Contribution	BPP	HKS	KN	MV	PdRV	Nyffeler
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	-19 ± 19	-19 ± 13
π, K loops $+$subl. N_C	–	–	–	0 ± 10	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	0	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	105 ± 26	116 ± 40

- Value $(80 \pm 40) \times 10^{-11}$ for KN was not given in Knecht + Nyffeler '01, but represents estimates used by the Marseille group before the appearance of MV '03. Reviews by Bijnens, Prades and Miller, de Rafael, Roberts from 2007 proposed $(110 \pm 40) \times 10^{-11}$.
- PdRV = Prades, de Rafael, Vainshtein '09: New combination of existing results (sometimes shifted, enlarged error). Do not consider dressed quark loop as separate contribution ! Assume that it is already taken into account by using short-distance constraint of MV '03 on pseudoscalar-pole contribution.
- The evaluation of the axial vectors by Melnikov + Vainshtein '03 is definitely some improvement over earlier calculations. It seems, however, again to be only the pole contribution. Nevertheless, we have taken over that value in our estimate.
- Added all errors linearly, rounded up $\pm 39 \times 10^{-11}$. PdeRV add errors in quadrature !

Summary of contributions to a_μ

- Leptonic QED contributions: $a_\mu^{\text{QED}} = (116\,584\,718.10 \pm 0.15) \times 10^{-11}$
- Electroweak contributions: $a_\mu^{\text{EW}} = (\underbrace{153.2}_{?} \pm \underbrace{1.8}_{?}) \times 10^{-11}$
- Hadronic contributions:
  - Vacuum Polarization: $a_\mu^{\text{had. v.p.}}(e^+e^-) = (\underbrace{6903.0}_{??} \pm \underbrace{52.6}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$

    $a_\mu^{\text{had. v.p.}}(\tau) = (\underbrace{7110}_{??} \pm \underbrace{58}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$
  - Light-by-Light scattering: $a_\mu^{\text{LbyL}} = (\underbrace{116 \pm 40}_{??}) \times 10^{-11}$
- Total SM contribution: $a_\mu^{\text{SM}}(e^+e^-) = (116\,591\,790.0 \pm 52.6 \pm 40 \pm 1.8\ [\pm 66.2]) \times 10^{-11}$

  $a_\mu^{\text{SM}}(\tau) = (116\,591\,997.0 \pm \underbrace{58}_{\text{v.p.}} \pm \underbrace{40}_{\text{LbyL}} \pm \underbrace{1.8}_{\text{QED + EW}}\ [\pm 70.5]) \times 10^{-11}$
- Experimental value: $a_\mu^{\text{exp}} = (116\,592\,080.0 \pm 63.0) \times 10^{-11}$

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) = (290 \pm 92) \times 10^{-11} \qquad [3.2\,\sigma]$$

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(\tau) = (83 \pm 95) \times 10^{-11} \qquad [0.9\,\sigma]$$

τ-data: Evaluation problematic because of isospin violations
e^+e^--data: Discrepancy real ? Sign for New Physics ?
Probably more work is needed in Theory and Experiment !

5. Conclusions

- Jegerlehner '07, '08: one should use off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ to evaluate pion-exchange contribution.
Prescription by Melnikov + Vainshtein '03 to use a constant (WZW) form factor at the external vertex only yields pion-pole contribution with on-shell form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$.
- Following this observation, we derived a new short-distance constraint at external vertex, relating the off-shell form factor to quark condensate magnetic susceptibility χ.
Problem: value of $\chi(\mu)$ and relevant scale μ not precisely known.
- We then performed new evaluation of pion-exchange contribution within large-N_C approximation using form factors that fulfill all QCD short-distance constraints.
Framework with two multiplets of vector resonances (LMD+V) for form factor and two-point function Π_{VT}.
Important imputs: $\chi = -(3.3 \pm 1.1)$ GeV^{-2} (via $\chi^{\mathrm{LMD}} = -2/M_V^2$), $h_6 = (5 \pm 5)$ GeV^4 (via $A^{\mathrm{LMD}}_{V,(p+q)^2}$).
Result for π^0:
$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$
- With updated values for η and η' (using simple VMD form factor):
$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$
- Combined with evaluations of the other contributions we get:
$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$$

More work is needed ! Soon first estimate from Lattice QCD ?

Backup slides

Hadronic light-by-light scattering: ChPT approach

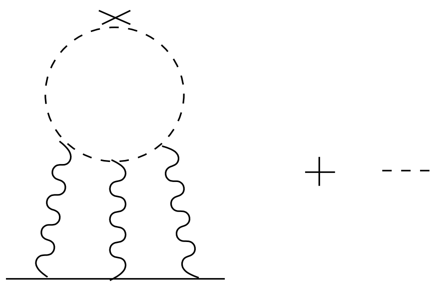
EFT for $E \ll 1$ GeV with pions, photons and muons

[de Rafael '94; M. Knecht, A.N., M. Perrottet, E. de Rafael, '02; Ramsey-Musolf + Wise '02]

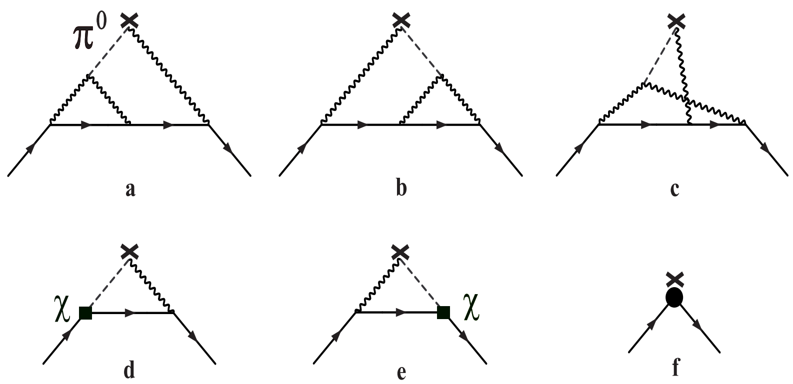
Note: chiral counting here refers to contribution to a_μ. Differs from counting in de Rafael '94 !

Contributions to $a_\mu^{\text{LbyL;had}}$

$\mathcal{O}(p^6)$: charged pion loop
(finite, subleading in $1/N_C$)



$\mathcal{O}(p^8)$: pion-pole (leading in $1/N_C$)



Divergent 2-loop contribution

$\longrightarrow$ need counterterms

1. One-loop graphs with insertion of χ
   (■) = coupling $\overline{\psi}\,\gamma_\mu\gamma_5\,\psi\,\partial^\mu\pi^0$

2. Local counterterm (●)

$\Rightarrow a_\mu^{\text{LbyL;had}}$ cannot be obtained in (pure) EFT framework
$\longrightarrow$ resonance models for form factors

Hadronic light-by-light scattering: Large log's

Renormalization group in EFT $\Rightarrow$ leading "large" logarithm $\ln^2(\mu_0/m_\mu)$

$$a_\mu^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) \right.$$ (loops with pions and kaons)

$$\left. + \overbrace{N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2}\frac{N_C}{3}\right)}^{\mathcal{C}\approx 0.025 \text{ (universal)}} \left[\ln^2\frac{\mu_0}{m_\mu} + \overbrace{c_1}^{\chi(\mu_0)\ (\pi^0\to e^+e^-)} \ln\frac{\mu_0}{m_\mu} + c_0\right] + \ldots \right\}$$

$f = -0.038$; $\mu_0 \sim M_\rho$: hadronic scale, $\ln\frac{M_\rho}{m_\mu} \sim 2$

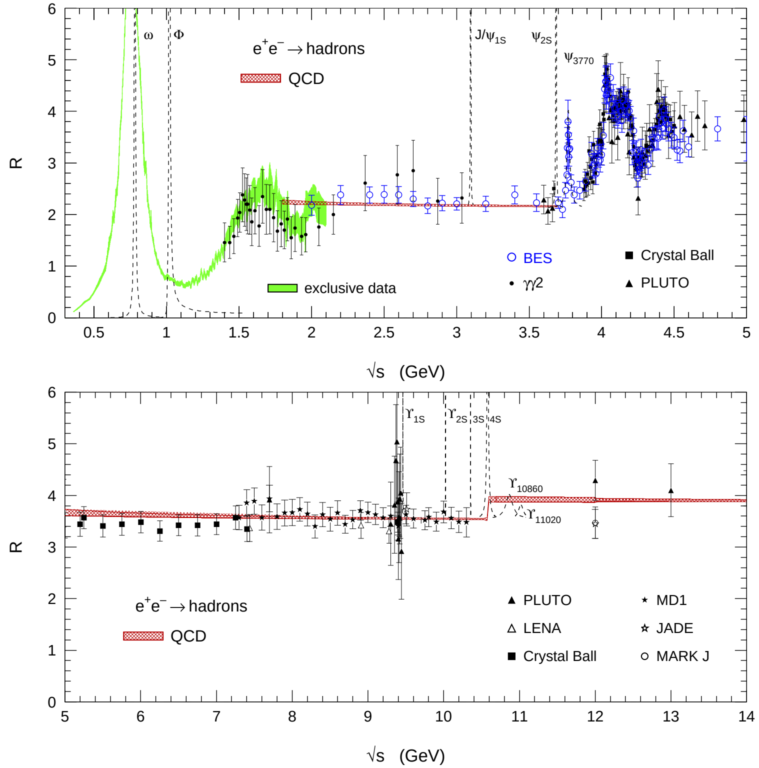
Problem: π^0-exchange $\longrightarrow$ cancellation between $\ln^2$ and $\ln$:

$$\begin{aligned} a_\mu^{\text{LbyL};\pi^0}\Big|_{\text{VMD}} &= \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \left[\ln^2\frac{M_\rho}{m_\mu} + c_1 \ln\frac{M_\rho}{m_\mu} + c_0\right] \\ &\overset{\text{Fit}}{=} \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \; [3.94 - 3.30 + 1.08] \\ &= [12.3 - 10.3 + 3.4] \times 10^{-10} \\ &= 5.4 \times 10^{-10} \end{aligned}$$

The Large-N_C World

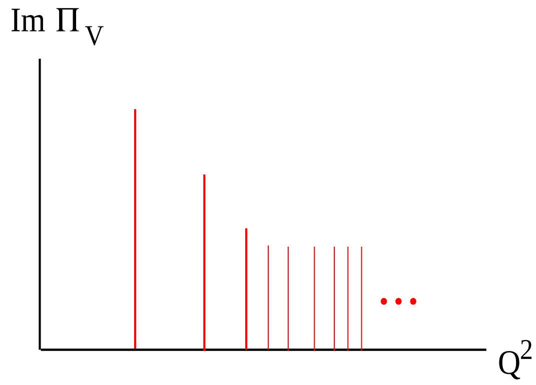
2-point function $\langle VV \rangle \longrightarrow$ spectral function Im$\mathbf{\Pi}_V \sim \sigma(e^+e^- \longrightarrow$ hadrons)

Real world

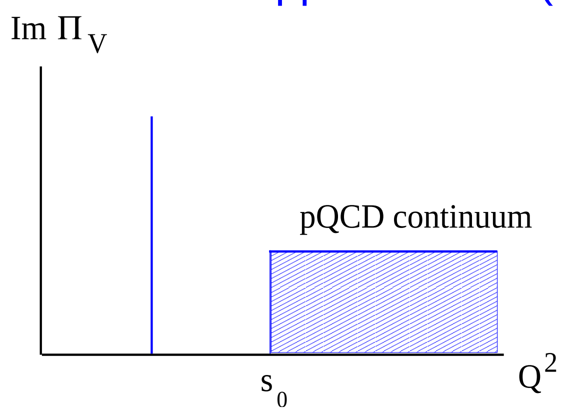


Davier et al., ’03

Large-N_C QCD (’t Hooft ’74)



Minimal Hadronic Approximation (MHA)



Scale s_0 fixed by the OPE

The Large-N_C World (cont.)

Adler function (Minimal Hadronic Approximation)

$$\mathcal{A}(Q^2) \equiv -Q^2 \frac{\partial \Pi_V(Q^2)}{\partial Q^2}$$

$$\mathcal{A}(Q^2)\Big|_{\text{MHA}} = \left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right) e^2 \left\{ 2 f_V^2 M_V^2 \frac{Q^2}{(Q^2+M_V^2)^2} + \frac{N_C}{16\pi^2}\frac{4}{3}\frac{Q^2}{Q^2+s_0}(1+\ldots) \right\}$$

Chiral loops (two-pion states) subleading in $1/N_C$

No $1/Q^2$ term in the OPE $\Rightarrow$ fixes s_0: $\quad 2 f_V^2 M_V^2 = \frac{N_C}{16\pi^2}\frac{4}{3} s_0 \left(1 + \frac{3}{8}\frac{\alpha_s(s_0)}{\pi} + \ldots\right)$

General relation:

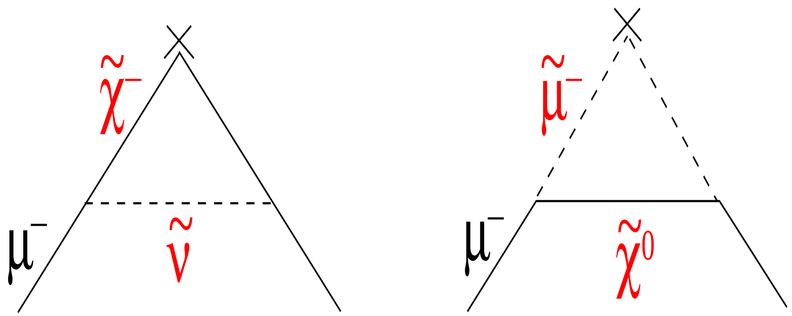
$$a_\mu^{\text{had. v.p.}} = \frac{\alpha}{\pi}\int_0^1 \frac{dx}{x}(1-x)\left(1-\frac{x}{2}\right)\mathcal{A}\left(\frac{x^2}{1-x} m_\mu^2\right)$$

$$a_\mu^{\text{had. v.p.}}\Big|_{\text{MHA}} = (5700 \pm 1700) \times 10^{-10}$$ (30% systematic error)

Compare with evaluation using experimental data (Davier et al. ’03)

$$a_\mu^{\text{had. v.p.}} = (6963 \pm 72) \times 10^{-10}$$

SUSY contributions to a_μ



Chargino $\tilde{\chi}^-$ contribution dominates over neutralino $\tilde{\chi}^0$

Large $\tan\beta$ limit (Czarnecki + Marciano '01):

$$\left|a_\mu^{\text{SUSY}}\right| \approx \frac{\alpha(M_Z)}{8\pi\sin^2\theta_W}\frac{m_\mu^2}{M_{\text{SUSY}}^2}\tan\beta\left(1-\frac{4\alpha}{\pi}\ln\frac{M_{\text{SUSY}}}{m_\mu}\right)$$
$$\approx 130\times10^{-11}\left(\frac{100\text{ GeV}}{M_{\text{SUSY}}}\right)^2\tan\beta$$

Compare: $a_\mu^{\text{EW}} \approx 150\times10^{-11}$

To explain $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) \approx 290\times10^{-11}$

$$\Rightarrow M_{\text{SUSY}} \approx 135-425\text{ GeV} \qquad (4<\tan\beta<40)$$