Aspects of the m_s chiral expansion and πK sum rules

Coll. with:

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Outline

* Introduction
* πK amplitude: matching at $O(p^4)$
* πK amplitude: matching at $O(p^6)$
* πK amplitude and πK vector/scalar form factors

Introduction

* At first Nambu : spontaneous symmetry breaking $\Rightarrow M_\pi$ small, Goldberger-Treiman relation: 2% accuracy !)

* Proofs in QCD: t'Hooft (1979), Vafa, Witten (1981)

* QCD $\longrightarrow$ Effective Lagrangian: Weinberg (1979) like QED, not one but several (growing number) coupling constants

* Systematic exploration, probe theory against experiment Gasser, Leutwyler (1984,1985)(25th anniversary)

* Expansion parameters: Energy (must be small), quark masses

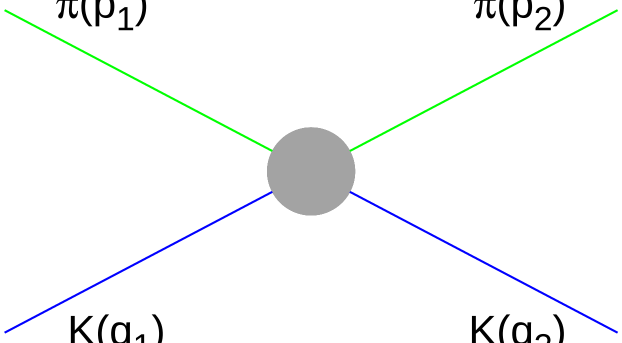
* Three quarks are light (compared to 1 GeV)
  - $m_u \, m_d << m_s \Longrightarrow$ two expansions
  - $N_f = 2$ exciting recent results ($\longrightarrow$ talks G. Colangelo, J. Gasser)
  - $N_f = 3$ Is there a problem ?

    Exemple: slope parameter (α) in $\eta \rightarrow 3\pi^0$

NNLO calc.	$\rightarrow \alpha = +0.013$	Bijnens, Ghorbani (2007)
Experiment	$\rightarrow \alpha = -0.027 \pm 0.010$	KLOE (2007)

πK amplitude: matching at $O(p^4)$

πK scattering



Mandelstam variables: s, t, u

$s = (p_1 + q_1)^2$, $t = (p_1 - p_2)^2$, $u = (p_1 - q_2)^2$

$s + t + u = 2m_K^2 + 2m_\pi^2$

Isospin symmetry assumed. Introduce

$$F^+(s,t,u) = \tfrac{1}{3}F^{\frac{1}{2}}(s,t,u) + \tfrac{2}{3}F^{\frac{3}{2}}(s,t,u)$$
$$F^-(s,t,u) = \tfrac{1}{3}F^{\frac{1}{2}}(s,t,u) - \tfrac{1}{3}F^{\frac{3}{2}}(s,t,u)$$

Crossing relations

$$s \leftrightarrow u \qquad F^\pm(s,t,u) = \pm F^\pm(u,t,s)$$
$$s \leftrightarrow t \qquad F^+(s,t,u) = \tfrac{1}{\sqrt{6}}G^0(t,s,u) \qquad \pi\pi \rightarrow K\overline{K}$$
$$F^-(s,t,u) = \tfrac{1}{2}G^1(t,s,u)$$

πK amplitude in ChPT at $O(p^4)$

* Calculation by V. Bernard, N. Kaiser, U. Meissner (1991)

$$F^+(s,t) = \left[\overline{W}_0^+(s) + (t-u)\overline{W}_1^+(s) + (s \leftrightarrow u)\right] + \overline{U}_0(t)$$
$$+\lambda_1^+ t^2 + \lambda_2^+(s-u)^2 + \beta^+ t + \alpha^+$$
$$F^-(s,t) = \left[\overline{W}_0^-(s) + (t-u)\overline{W}_1^-(s) - (s \leftrightarrow u)\right] + (s-u)\overline{U}_1(t)$$
$$+(s-u)(\lambda_1^- t + \beta^-)\ .$$

* Where: $\overline{W}_i^{\pm}(z)$, $\overline{U}_i(z)$ = loop functions

$$\bar{J}_{PQ}(s) = \frac{s}{16\pi^2}\int_{(m_P+m_Q)^2}^{\infty} ds' \frac{\sqrt{(s'-(m_P+m_Q)^2)(s'-(m_P-m_Q)^2)}}{(s')^2(s'-s)}$$

$PQ = \pi K,\ \eta K, \pi\pi,\ K\overline{K},\ \eta\eta$

* Polynomial coefficients:

$$F_\pi^2 \beta^- = \frac{1}{4} + \frac{2m_\pi^2}{F_\pi^2}\Big[L_5^r(\mu) + \log' s\Big]$$

$$F_\pi^4 \lambda_1^- = -L_3^r + \log' s\Big]$$

$$\alpha^+ = \frac{8m_\pi^2 m_K^2}{F_\pi^4}\Big[4L_1^r + L_3^r - 4L_4^r - L_5^r + 4L_6^r + 2L_8^r + \log' s\Big]$$

$$\beta^+ = \beta^- + \frac{8(m_\pi^2 + m_K^2)}{F_\pi^4}\Big[-2L_1^r - \frac{1}{2}L_3^r + L_4^r + \log' s\Big]$$

$$F_\pi^4 \lambda_1^+ = \Big[8L_1^r + 2L_2^r + \frac{5}{2}L_3^r + \log' s\Big]$$

$$F_\pi^4 \lambda_1^+ = \Big[2L_2^r + \frac{1}{2}L_3^r + \log' s\Big]$$

ChPT versus experiment for πK scattering

∗ **Difficulties:**

- No low energy data now (In near future: $D \to K\pi l\nu$, πK atoms...)
- Lack of exact unitarity in ChPT

∗ **Solution:** Use analyticity (true in QCD) of scattering amplitude

- Cauchy representations [dispersion relations] $\Longrightarrow$ amplit. in unphysical regions e.g. around $t = 0$, $s = u$ (subthreshold coefficients)
- Obtaining directly sum rules for L_i's B. Ananthanarayan, P. Büttiker (2001)

AB sum rules

∗ Idea:

Recast dispersion relations in same form as ChPT formula (Fuchs, Sazdjian, Stern (1993) for $\pi\pi$)

∗ Analyticity:

– Fixed t dispersion relations [rigorous proofs S. Mandelstam (1960), A. Martin (1966)]

– Fixed us dispersion relations [proof: Auberson]

∗ Large energy

– Froissart bound [rigorous]

– Regge pole phenomenology [not rigorous, well tested]

* **Chiral counting in dispersion relations**

– Introduce a scale $\Lambda \simeq 1$ GeV

$z < \Lambda^2$: $\mathrm{Im}\, f_{l\geq 2}(z)$, $\mathrm{Im}\, g_{l\geq 2}(z)$ = $O(p^8)$ [3 loops diagrams]

$\Longrightarrow$ can be dropped

$z > \Lambda^2$: expand in powers s/z, t/z, u/z

* Amplitude gets expressed with one-variable functions. Integrals:

$W^{\pm}_{0,1}(z) \longrightarrow [\pi K \to \pi K]_{l=0,1}$ partial-waves

range: $[(m_\pi + m_K)^2, \Lambda^2]$

$U_{0,1}(z) \longrightarrow [\pi\pi \to K\overline{K}]_{l=0,1}$ partial-waves

range: $[4m_\pi^2, \Lambda^2]$

* Chiral – Dispersive: expanded as $s,\ t$ polynomial and set =0

$\Rightarrow$ sum rules. Used with $O(p^4)$. ($O(p^6)$ possible in principle)

Experimental inputs

* Available at present

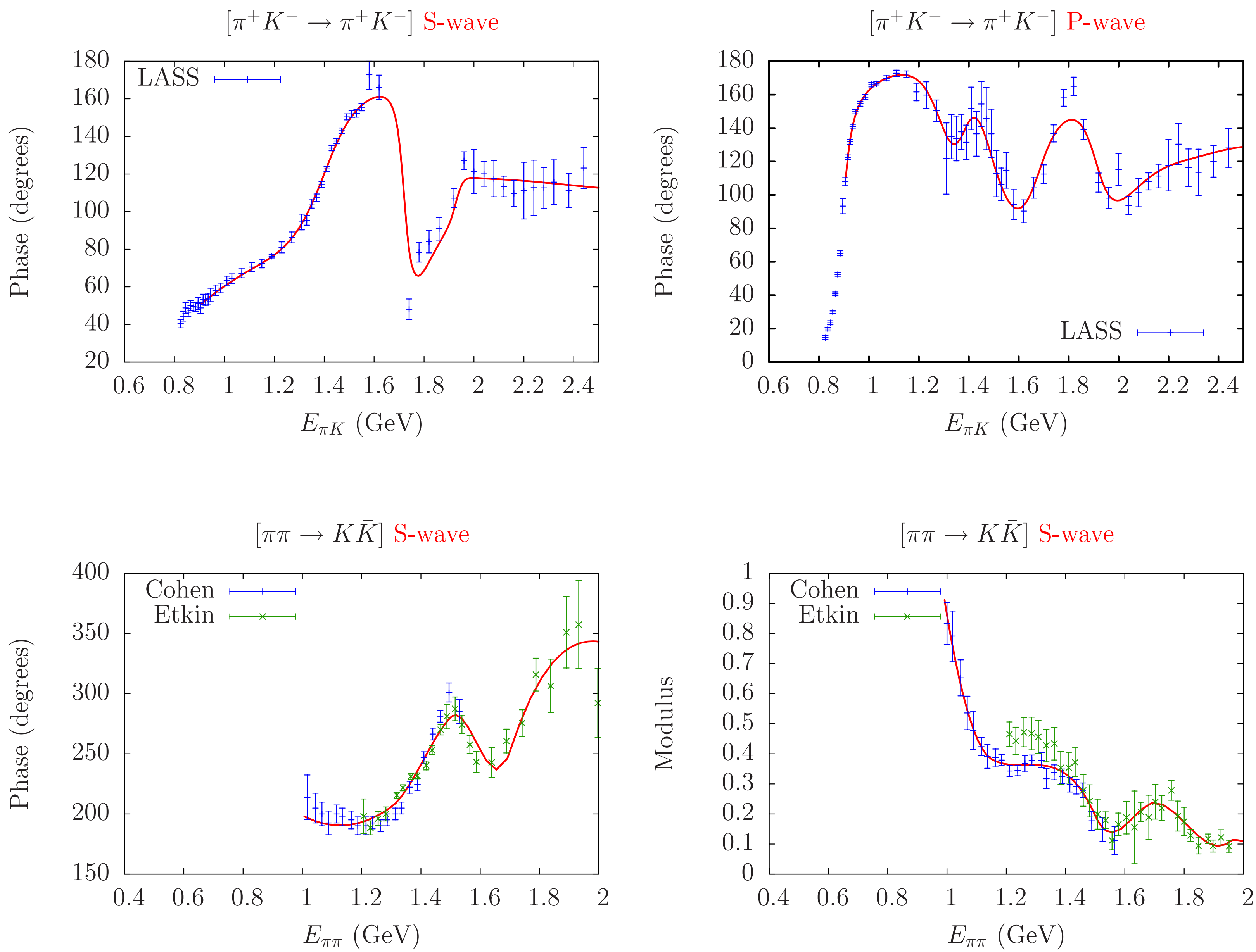
* From production experiments: $Kp \to K\pi p$
  High statistics (Estabrooks (1978), Aston (LASS) (1988))

  $\pi K \to \pi K$ Partial waves: $l = 0, 1, 2, 3, 4, 5$
  Energy range: $0.8 \text{ GeV} \lesssim E \lesssim 2.5 \text{ GeV}$

* From production experiments: $\pi p \to K\overline{K}p$
  (Cohen (1980), Etkin (1982))

  $\pi\pi \to K\overline{K}$ Partial waves: $l = 0, 1, 2$
  Energy range: $1.0 \text{ GeV} \lesssim E \lesssim 2.0 \text{ GeV}$

* Illustration of some experimental inputs:



Amplitudes in low energy regions

* Constrained by analyticity

* PLUS key ingredient elastic unitarity

(holds exactly in some region as π, K are lightest hadrons)

* One may assume

1) $\pi K \to \pi K$ elastic up to $\sim 1\mathrm{GeV}$
2) $\pi\pi \to \pi\pi$ elastic up to $K\bar{K}$ threshold

(not exact but good approximation)

* Then for $\pi\pi \to K\bar{K}$

$$\mathrm{Im}\, g_l^I(t) = \sqrt{\frac{s - 4m_\pi^2}{s}} \left(g_l^I(t)\right)^* \underbrace{h_l^I(t)}_{\pi\pi \to \pi\pi \text{ amplitude}}$$

* Together with dispersive representation

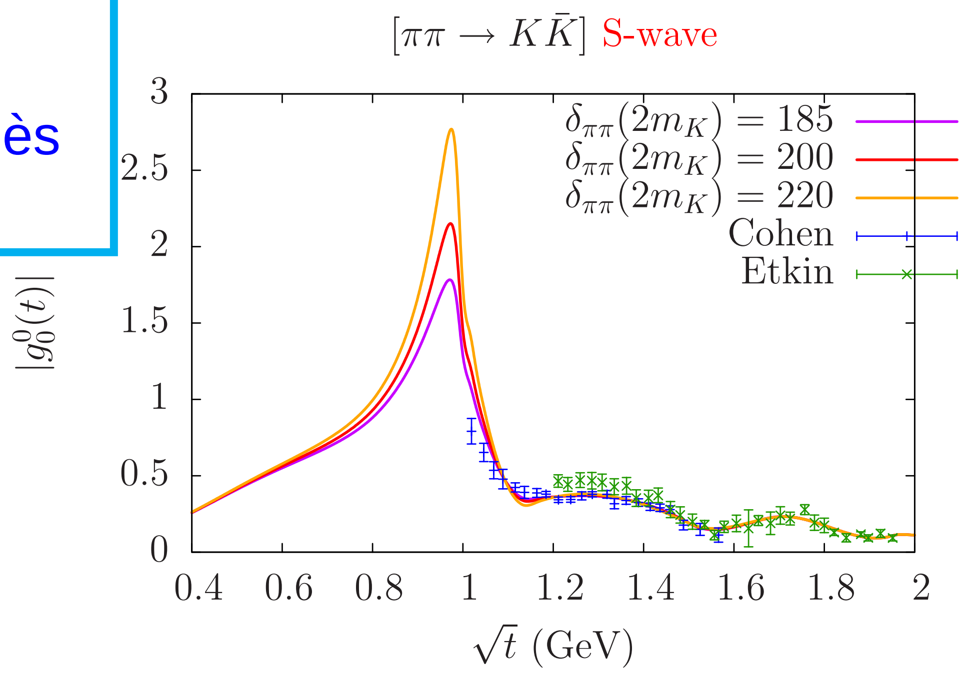
$$g_0^0(t) = g_0^0(0) + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}\, g_0^0(t')}{t'(t'-t)} dt' + \underbrace{\Delta(t)}$$

left-hand cut

* $g_0^0(t)$, $g_1^1(t)$ in unphysical region $4m_\pi^2 \leq t \leq 4m_K^2$:

Phase:	= $\pi\pi \to \pi\pi$ phase (Watson's theorem)
Modulus:	Muskhelishvili-Omnès formulas

Done in: Ananthanarayan, P. Büttiker, B.M. (2001)



Roy-Steiner equations

* Sum rules can now be evaluated

* Further refinement possible in low energy region:

Treat $\pi K \to \pi K$ I=0,1 $(f_0^{\frac{1}{2}},\ f_0^{\frac{3}{2}},\ f_1^{\frac{1}{2}})$

$\pi\pi \to K\overline{K}$ I=0,1 (g_0^0, g_1^1)

In fully self-consistent way

* Encoded in Roy-Steiner equations Roy (1971) (for $\pi\pi$), Steiner (1971) (for πN, πK)

* Typical RS equation

$$\mathrm{Re}\, f_0^{\frac{1}{2}}(s) = \frac{1}{2} m_+ a_0^{\frac{1}{2}} + \frac{1}{12} m_+ (a_0^{\frac{1}{2}} - a_0^{\frac{3}{2}}) \frac{(s-m_+^2)(5s+3m_-^2)}{(m_+^2 - m_-^2)\, s}$$
$$+ \frac{1}{\pi}\int_{m_+^2}^{\infty} ds' \sum_{l=0}^{\infty} \left\{ K_{0l}^{\frac{1}{2}}(s,s') \mathrm{Im}\, f_l^{\frac{1}{2}}(s') + K_{0l}^{\frac{3}{2}}(s,s') \mathrm{Im}\, f_l^{\frac{3}{2}}(s') \right\}$$
$$+ \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \sum_{l=0}^{\infty} \left\{ K_{02l}^{0}(s,t') \mathrm{Im}\, g_{2l}^{0}(t') + K_{02l+1}^{1}(s,t') \mathrm{Im}\, g_{2l+1}^{1}(t') \right\}$$

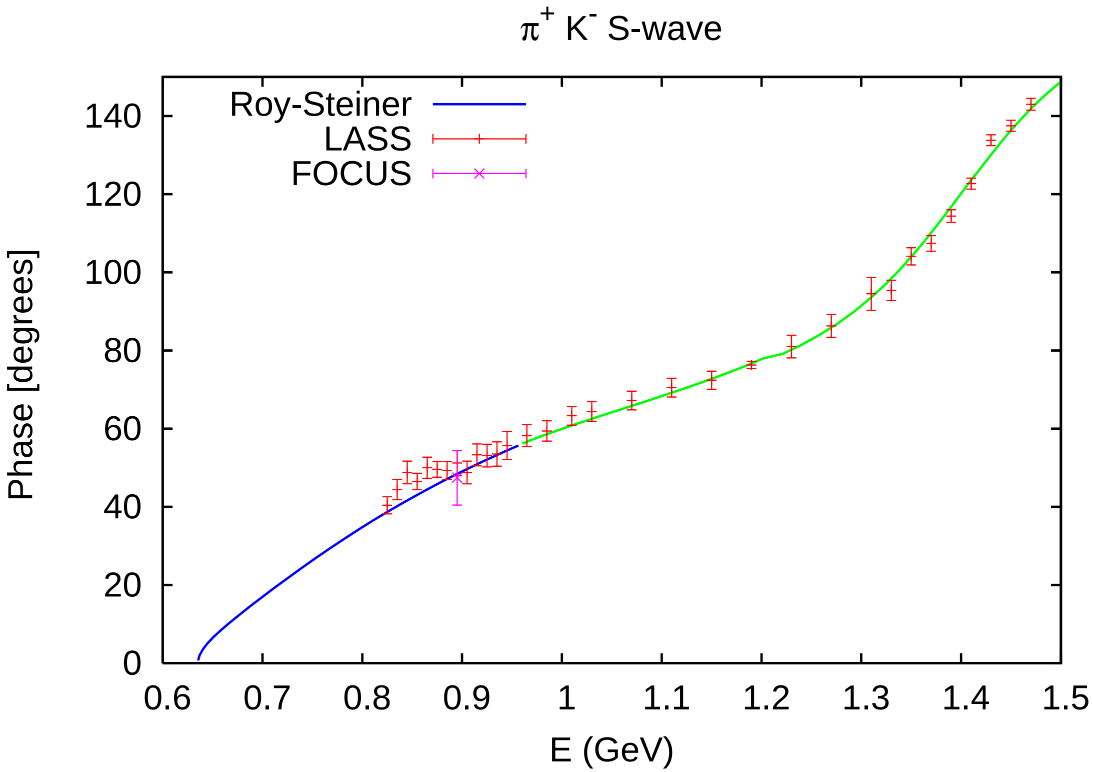
where:
- a_0^I: $\pi K \to \pi K$ scattering lengths
- $f_l^I(s')$: $\pi K \to \pi K$ PW amplitude
- $g_l^I(t')$: $\pi\pi \to K\overline{K}$ PW amplitude

* Combined with elastic unitarity

$$\mathrm{Im}\, f_0^{\frac{1}{2}}(s) = \sqrt{\frac{(s-m_+^2)(s-m_-^2)}{s^2}} |f_0^{\frac{1}{2}}(s)|^2$$

∗ One obtains a closed system of equations

∗ Wander's theorem: solution is unique (with proper boundary conditions)



∗ Numerical solutions: Büttiker, Descotes-Genon, B.M. (2004)

Matching at $O(p^4)$

1) AB sum rules: chiral constraints used in sum rules, also at $t = 0$, $s = (m_K + m_\pi)^2$ for low-energy extrapolation

2) Subthreshold sum rules: RS solution control low energy regions

$\ast$ Subthreshold parameters:

Expand around $t = 0, \; s = u = m_K^2 + m_\pi^2$

$$F^+(s,t,u) = C_{00}^+ + C_{10}^+ \, t + C_{20}^+ \, t^2 + C_{01}^+ \left(\frac{s-u}{4m_K}\right)^2 + \ldots$$

$$F^-(s,t,u) = \left(\frac{s-u}{4m_K}\right) \left[C_{00}^- + C_{10}^- \, t + \ldots\right]$$

* Results: $L_1,\ L_2,\ L_3,\ L_4$: good precision

  $L_5,\ L_8 + 2L_6$ poor precision

* Comparison with other determinations:

  – K_{l4} form factors: use $\frac{f'_s(0)}{f_s(0)},\ \frac{g_p(0)}{f_s(0)},\ \frac{g'_p(0)}{f_s(0)}$ from NA48/2, Bloch-Devaux, (2008)

  – $\pi\pi$ sum rules Colangelo, Gasser, Leutwyler (2001)

$10^3\ L_1^r$	$10^3\ L_2^r$	$10^3\ L_3^r$	$10^3\ L_4^r$	$\mu = 0.77$ GeV
0.84 ± 0.15	1.36 ± 0.13	-3.65 ± 0.45	0.22 ± 0.45	πK (1)
1.05 ± 0.12	1.32 ± 0.03	-4.53 ± 0.15	0.53 ± 0.39	πK (2)
0.59 ± 0.24	1.27 ± 0.84	-3.01 ± 1.43	$-$	K_{l4} (NA48/2)
0.62 ± 0.02	1.24 ± 0.05	-4.03 ± 0.09	$-$	$\pi\pi$ (+large N_c)

* Comparison of L_4 with lattice QCD

$10^3\, L_4^r$	$\mu = 0.77$ GeV	
0.22 ± 0.45	πK (1)	
0.53 ± 0.39	πK (2)	
$0.3(3)\, \binom{+3}{-1}$	MILC	[0710.1118]
0.34 ± 0.18	UKQCD	[0804.0473]

* Uncertainty related to $\pi\pi \to K\overline{K}$ extrapolation

* Lattice determ. method different from ours

* Physics of L_4: $N_F = 2$ versus $N_F = 3$

$$F_{(3)} = \lim (F_\pi)_{m_u=m_d=m_s=0}$$
$$F_{(2)} = \lim (F_\pi)_{m_u=m_d=0}$$

* ChPT $O(p^4)$ relation:

$$F_{(2)} = F_{(3)} \left[1 + \frac{m_s B_0}{F_{(3)}^2} \left(8L_4^r(\mu) - \frac{1}{32\pi^2} \log \frac{m_s B_0}{\mu^2} \right) \right]$$

* $F_{(3)}$ reduced compared with $F_{(2)}$

$$\frac{F_{(2)}}{F_{(3)}} - 1 \simeq 10 - 20\%$$

πK amplitude: matching at $O(p^6)$

Amoros-Bijnens-Talavera strategy:

* Many computations at NNLO (i.e $O(p^6)$) performed in $N_F = 3$ ChPT, partial list:

Amoros, Bijnens, Talavera (1999)	VV, AA, masses, decay constants
Amoros, Bijnens, Talavera (2000)	condensates, K_{l4}
Bijnens, Talavera (2002)	Electromagnetic form factors
Bijnens, Talavera (2003)	K_{l3} form factors
Bijnens, Dhonte (2003)	Scalar form factors
Bijnens, Dhonte (2004)	$\pi\pi$ scattering
Bijnens, Dhonte, Talavera (2004)	πK scattering

* $O(p^6)$ Lagrangian contains 90 coupling constants: $C_1, ..., C_{90}$

* Strategy: Use naive resonance saturation model to estimate $C_1, ..., C_{90}$, refit $L_1, ..., L_{10}$ then predict

* Naive resonance saturation model was shown to reproduce reasonably well the p^4 couplings L_i (Ecker, Gasser, Pich, de Rafael (1989))

* Method: Introduce a Lagrangian containing vector and scalar resonances + couplings to U and sources

* Integrating out the resonances one gets:

$$L_1 = \frac{G_V^2}{8M_V^2} - \frac{c_d^2}{8M_S^2}, \quad L_2 = \frac{G_V^2}{4M_V^2}, \quad L_3 = -\frac{G_V^2}{2M_V^2} + \frac{c_d^2}{2M_S^2},$$

$$L_4 = \frac{\tilde{c}_d \tilde{c}_m}{M_{S_1}^2}, -\frac{c_d c_m}{3M_S^2}, \quad L_5 = \frac{c_d c_m}{M_S^2}, \quad L_6 = \frac{\tilde{c}_d^2}{2M_{S_1}^2} - \frac{c_m^2}{6M_S^2}$$

etc...

* Agreement with experimental $L_i^r(\mu)$ with scale $\mu = M_\rho$

Comparing $O(p^6)$ predictions and experiment

$*$ (Some) predictions of Bijnens, Dhonte (2004) for πK subthreshold coefficients disagree with sum rules

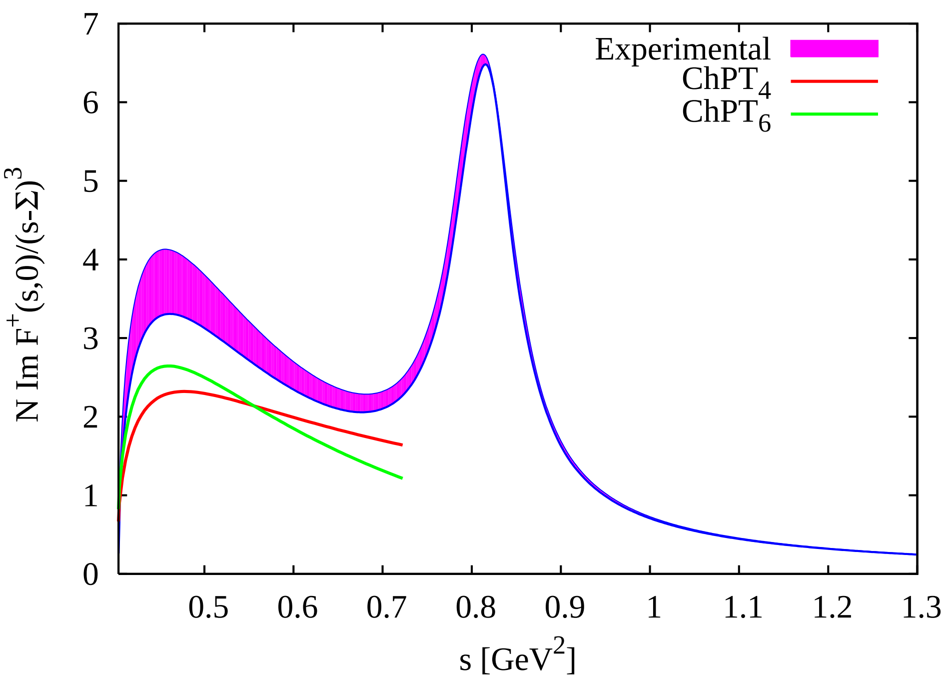
	Bijnens-Dhonte	Sum rules
C_{20}^{+}	0.003	0.024 ± 0.006
C_{01}^{+}	3.8	2.07 ± 0.10
C_{10}^{-}	0.09	0.31 ± 0.01

$*$ What could be the reason ?

∗ Details on sum rules. One exemple:

$$C^+_{01} = \frac{8m_K^2 m_\pi^2}{\pi} \int_{(m_K+m_\pi)^2}^{\infty} \frac{\mathrm{Im}\, F^+(s',0)}{(s'-m_K^2-m_\pi^2)^3}\, ds'$$

∗ Integrand



∗ No problem with the sum rule

Improving the resonance model

* Resonance model of Amoros, Bijnens, Talavera (2000) has:

  8 parameters (same as Ecker et al. (1989))
  + 7 parameters.

* Vector resonances:

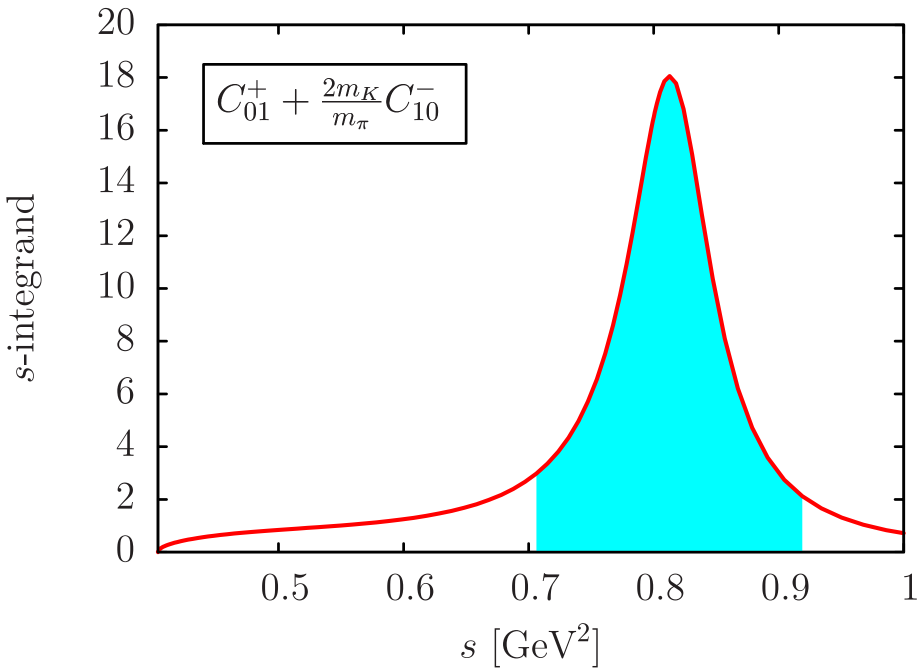
  Flavour symmetry breaking (1 param. f_χ) not enough:
  $\Gamma_{K^*(892)} - \Gamma_{\rho(770)} \neq 0$ but $m_{K^*(892)} - m_{\rho(770)}$=0

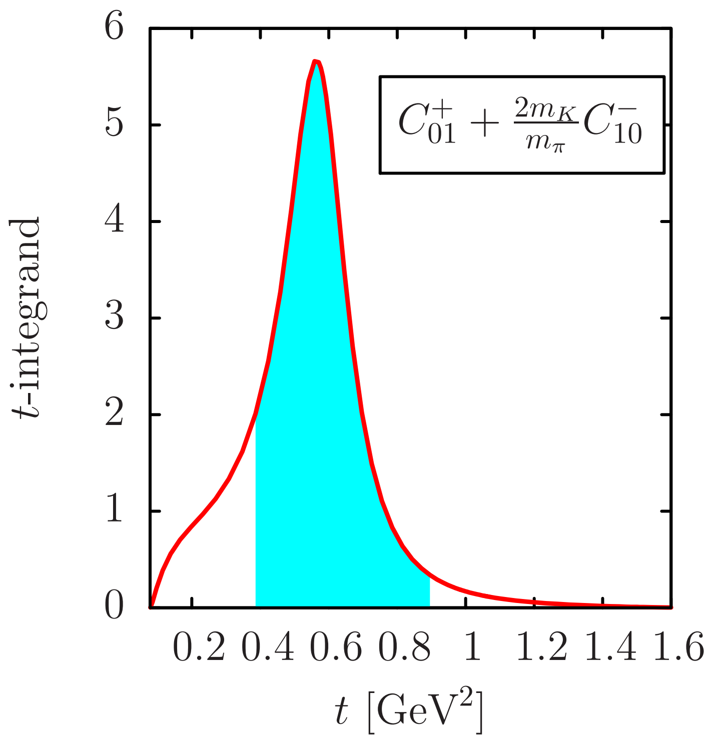
* Model with improved symm. breaking (K. Kampf, B.M. (2006)):

  5 parameters: $f_\chi,\ g^m_{V_1},\ g^m_{V_2},\ e^m_V,\ f^m_{V_1}$

  Determination: $\Gamma_{K^*(892)} - \Gamma_{\rho(770)}$, $m_{K^*(892)} - m_{\rho(770)}$
  $\pi(1300) \to \rho\pi$, $\rho \to K\overline{K}$, $F_{K^*(892)} - F_\rho$

* Resonance saturation of an observable quantity:





* Note: no S-wave contribution in combination $C_{01}^{+} + 2m_K/m_\pi C_{10}^{-}$

* Contibution from resonance regions $[C_{01}^{+} + 2m_K/m_\pi C_{10}^{-}]_V$:

Amoros model	Improved model	sum rule
0.74	3.58	3.50

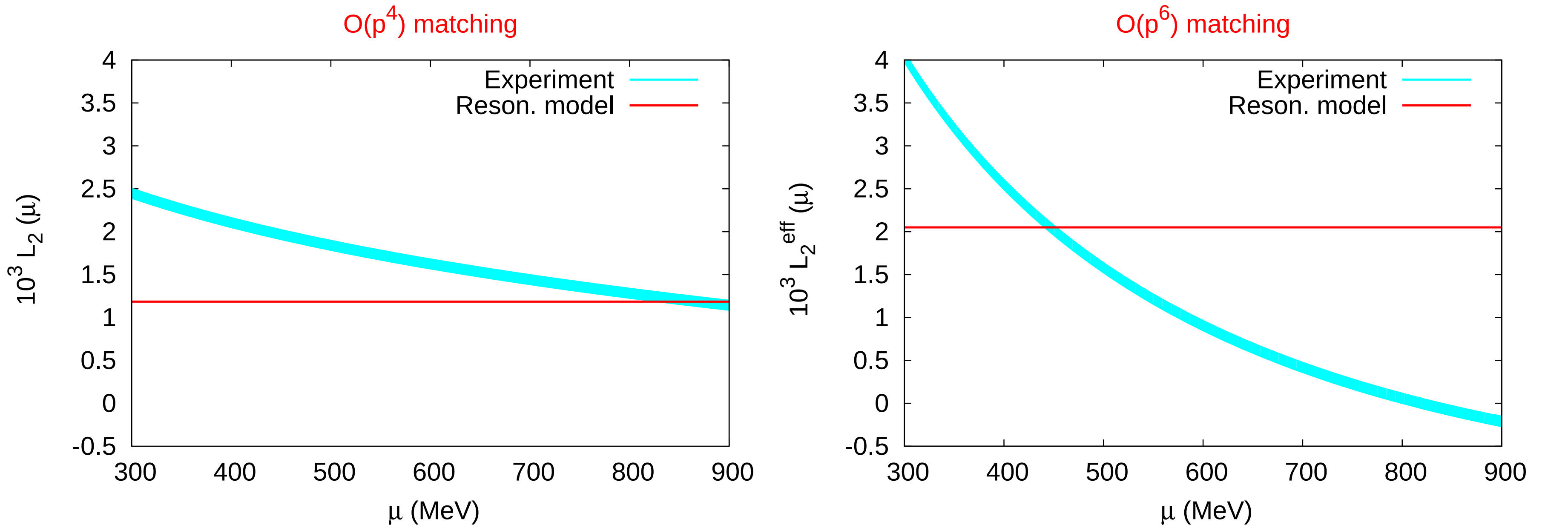
∗ Resonance saturation of a scale dependent quantity:

– Coupling constants determined from $C^{+}_{01} + 2m_K/m_\pi C^{-}_{10}$:

$$O(p^4): \longrightarrow \quad L_2^r(\mu)$$

$$O(p^6): \longrightarrow \quad L_2^{eff}(\mu) = L_2^r(\mu) + \frac{m_K^2+m_\pi^2}{F_\pi^2}\left(-2C_4^r + C_{10}^r - 2C_{12}^r + 2C_{22}^r + 2C_{23}^r - C_{25}^r\right) + \frac{4m_K^2+2m_\pi^2}{F_\pi^2}\left(C_{11}^r - 2C_{13}^r\right)$$

∗ **Results:**



∗ The moral: p^6 numerical predictions based on resonance sat. estimates of C_i (probably) unreliable

πK amplitude and πK vector/scalar form factors

Motivations

* τ/charm factories: Cabibbo suppressed τ decays, e.g. $\tau \to K\pi\nu_\tau$.
Belle, Babar already 10^3 better than Aleph

* $\tau \to K\pi\nu_\tau$: two form factors, πK scattering via final-state interaction

$$f_+^{K\pi}(t): \quad I = \tfrac{1}{2} \text{ P-wave}$$
$$f_0^{K\pi}(t): \quad I = \tfrac{1}{2} \text{ S-wave}$$

* Method: analyticity, unitarity (again)
also low energy + high energy constraints

* Goal: "deduce" form-factors from scattering information

* References:
Donoghue, Gasser, Leutwyler (1990) [$\pi\pi$ scalar form factor]
Jamin, Oller, Pich (2001) [πK scalar form factor]

* Determination of V_{us}:
  $V_{us} f_+^{K\pi}(0)$ measured in K_{l3} decays

* Bijnens, Talavera (2003) relation: [via couplings C_{12}, C_{34}]

$$f_+^{K\pi}(0) \quad \text{from} \quad \frac{d}{dt} f_0^{K\pi}(0),\ \frac{d^2}{dt^2} f_0^{K\pi}(0)$$

[Derivatives at $t = 0$ could be determined from Cauchy representation]

* Can one separate $f_0^{K\pi}(t)$, $f_+^{K\pi}(t)$ in τ decay experiment ?

* Here: method extended to $f_+^{K\pi}(t)$, probe against experiment
Belle (2007)

Basic properties

* $f_+^{K\pi}(t)$ analytic in t, cut $[(m_K + m_\pi)^2, \infty]$

* Asymptotically (Brodsky, Lepage (1980))

$$f_+^{K\pi}(-t)\Big|_{t\to\infty} = \frac{16\pi\sqrt{2}\,\alpha_s(t)F_\pi^2}{t}$$

$\Rightarrow$ Unsubtracted dispersion relation

$$f_+^{K\pi}(t) = \frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty} \frac{\mathrm{Im}\, f_+^{K\pi}(t')}{t'-t}\, dt'$$

Expressing the imaginary part

* Quite generally from

$$2i\mathrm{Im}\ \underset{out}{\langle}\ m|\mathcal{O}|0\rangle = \underset{out}{\langle}\ m|\mathcal{O}|0\rangle - \underset{out}{\langle}\ m|\mathcal{O}|0\rangle^*$$

* Using time-reversal invariance
  + definition of $S-$matrix

$$S_{mn} = \underset{out}{\langle}\ m|n\ \underset{in}{\rangle}$$

$\Rightarrow$ Linear relation in terms of T-matrix elements

$$\mathrm{Im}\ \underset{out}{\langle}\ m|\mathcal{O}|0\rangle = \frac{1}{2}\sum_n (T_{mn})^*\ \underset{out}{\langle}\ n|\mathcal{O}|0\rangle$$

∗ For given energy E of state $|m\rangle$ finite sum over $|n\rangle$

∗ Case of $f_+^{K\pi}(t)$:

$\sqrt{t} \lesssim 1$ GeV: $|n\rangle = |\pi K\rangle$

$\sqrt{t} \lesssim 2.5$ GeV: $|n\rangle = |\pi K\rangle, \ |\pi K^*\rangle, \ |\rho K\rangle$

∗ Measurements by LASS (1984), (1987), (1988) T-matrix in resonance regions:

	$K\pi$	$K^*\pi$	$K\rho$
$K^*(892)$	100	0.0	0.0
$K^*(1410)$	6.6 ± 1	> 40	< 7
$K^*(1680)$	38.7 ± 2.5	$29.9^{+2.2}_{-4.7}$	$31.4^{+4.7}_{-2.1}$

∗ Approximations a) simplified inelasticity b) truncation to three states extended to $\sqrt{t} > 2.5$ GeV

$\Rightarrow$ Form factors obey coupled channel Muskhelishvili-Omnès (singular) integral equations

Fits to πK scattering with 3x3 unitarity

* Unitarity enforced with simple $K-$matrix method

$$T^{-1} = K^{-1} - diag\left(\bar{J}_1(t), \bar{J}_2(t), \bar{J}_3(t)\right)$$

* Resonance contributions [poles in $K-$matrix]
  - Good results with four resonances $[K^*(890)$, $K^*(1410)$, $K^*(1680)$, $K^*(2200)]$
  - Amplitudes computed from resonance chiral Largrangian

$$\mathcal{L}_g^{(n)} - = \frac{-i}{2} g_V(n)\, \mathrm{tr}\,(V_{\mu\nu}^{(n)}[u_\mu, u_\nu])$$

$$\mathcal{L}_\sigma^{(n)} = \frac{1}{2}\sigma_V(n)\, \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(V_{\mu\nu}^{(n)}\{u_\rho, V_\sigma\})$$

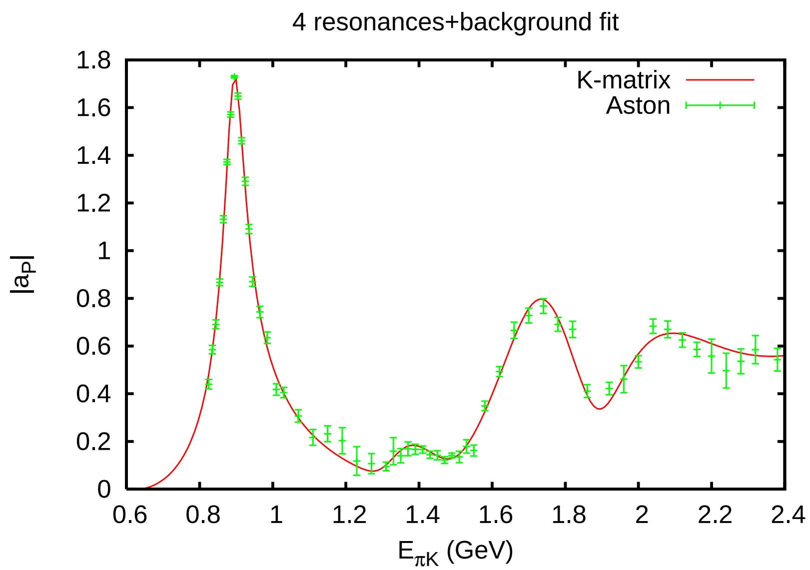
* Nonresonant contributions: simple parametrization

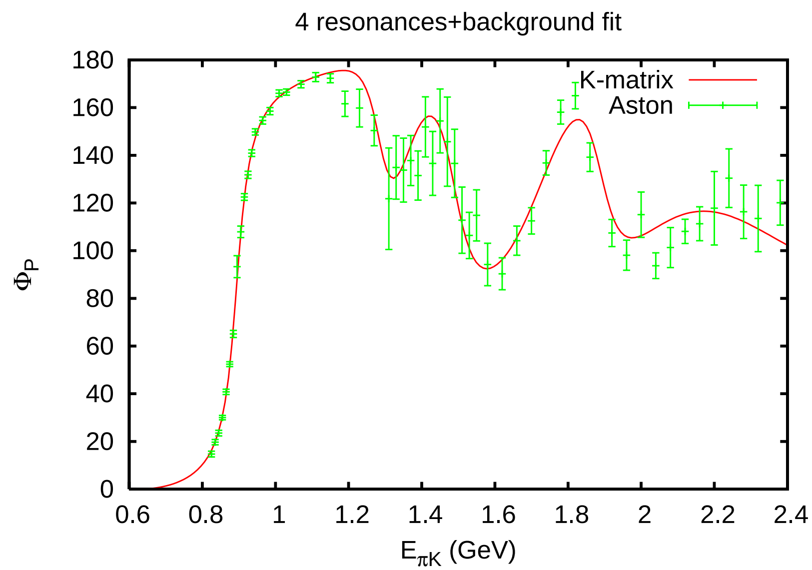
∗ Altogether 12+4 parameters

∗ Agreement with Prades (1993) for $g_V(1)$, $\sigma_V(1)$

	Fit	Prades(1993)
$g_V(1)$	0.073	0.083
$\sigma_V(1)$	0.26	0.25

∗ Good description of $\pi K \to \pi K$





$T-$matrix in asymptotic region

* Solving the Muskhelishvili-Omnès equations for the form factors

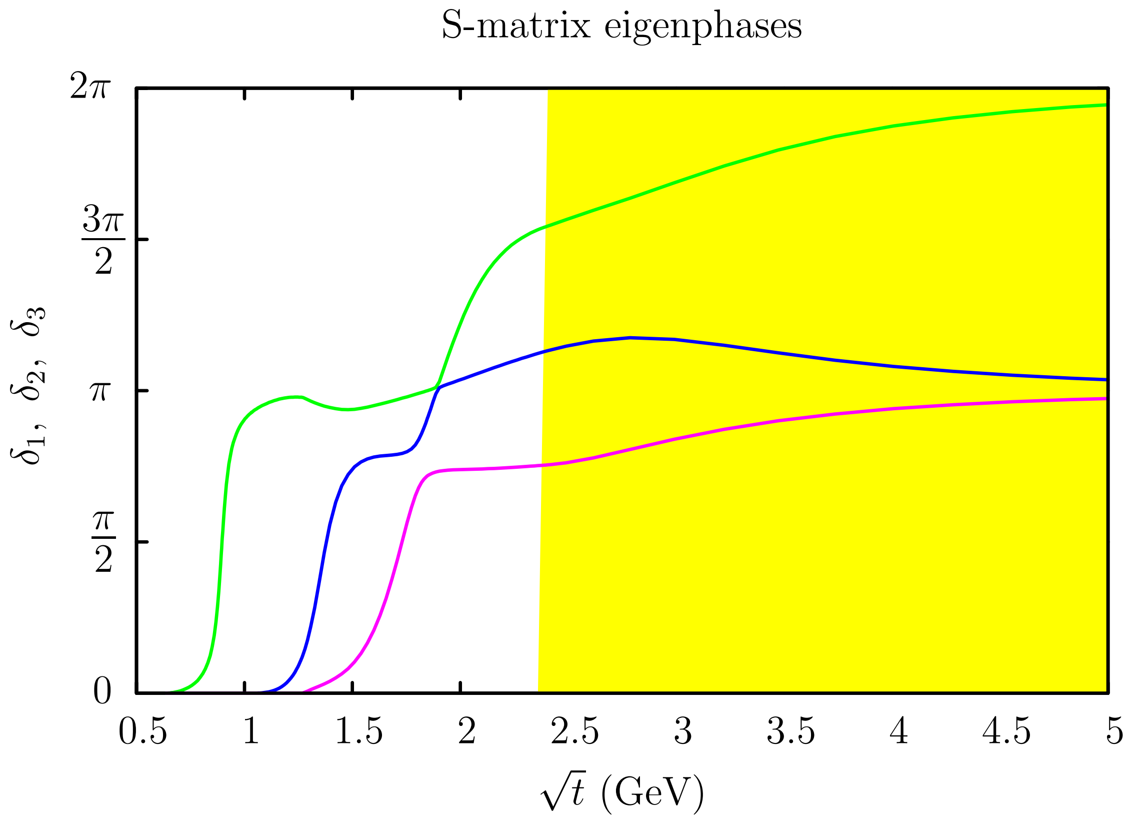
* Muskhelishvili theorem
  - $S-$matrix: $S(t) \equiv \exp(iD(t))$
  - Index: $N = \lim_{t\to\infty} \mathrm{tr}(D(t))/\pi$
  - N=number of independent solutions

* Experimentally:

$$\left.\frac{\mathrm{tr}(D(t))}{\pi}\right|_{t=6.25\ \mathrm{GeV}^2} \simeq 3.5$$

$\Rightarrow$ We choose $N = 4$

* Illustration of asymptotic interpolation [eigenvalues of D]



* impose four conditions: $f_+^{K\pi}(0)$, $H_2(0)$, $H_3(0)$ and $f_+^{K\pi}(\infty)$

∗ Definition of form factors:

$$\langle K^+(p_K)|\bar{u}\gamma^\mu s|\pi^0(p_\pi)\rangle = \frac{1}{\sqrt{2}}\left[f_+^{K\pi}(t)\,(p_K+p_\pi)^\mu + f_-^{K\pi}(t)\,(p_K-p_\pi)^\mu\right]$$

$$\langle K^{*+}(p_V,\lambda)|\bar{u}\gamma_\mu s|\pi^0(p_\pi)\rangle = \epsilon_{\mu\nu\alpha\beta}\, e^{*\nu}(\lambda) p_V^\alpha p_\pi^\beta\, H_2(t)$$

$$\langle \rho^0(p_V,\lambda)|\bar{u}\gamma_\mu s|K^-(p_K)\rangle = -\epsilon_{\mu\nu\alpha\beta}\, e^{*\nu}(\lambda) p_V^\alpha p_K^\beta\, H_3(t)$$

Conditions at $t = 0$ and $t = \infty$

$*$ $t = \infty$, Brodsky-Lepage +flavour symmetry
(used with constant $\alpha_s = 0.2$)

$*$ $t = 0$

- $f_+^{K\pi}(0) = 0.977 + O(m_s - \hat{m})^3$ [Gasser, Leutwyler (1985)]
- Flavour symmetry limit (+large N_c) from $\Gamma(\rho^+ \to \pi^+ \gamma)$.

$$H_2(0) = -H_3(0) = (1.54 \pm 0.08)\ \mathrm{GeV}^{-1}$$

- First order symmetry breaking:

$$H_2(0) = \quad (1.41 \pm 0.09 - 65.4\, a)\ \mathrm{GeV}^{-1}$$
$$H_3(0) = (-1.34 \pm 0.07 - 65.4\, a)\ \mathrm{GeV}^{-1}$$

- One undetermined parameter left : $a \sim O(10^{-3})$

Determination of a

* a can be determined from $\tau \to K\pi\nu_\tau$ decay rate

* New results slightly smaller than PDG

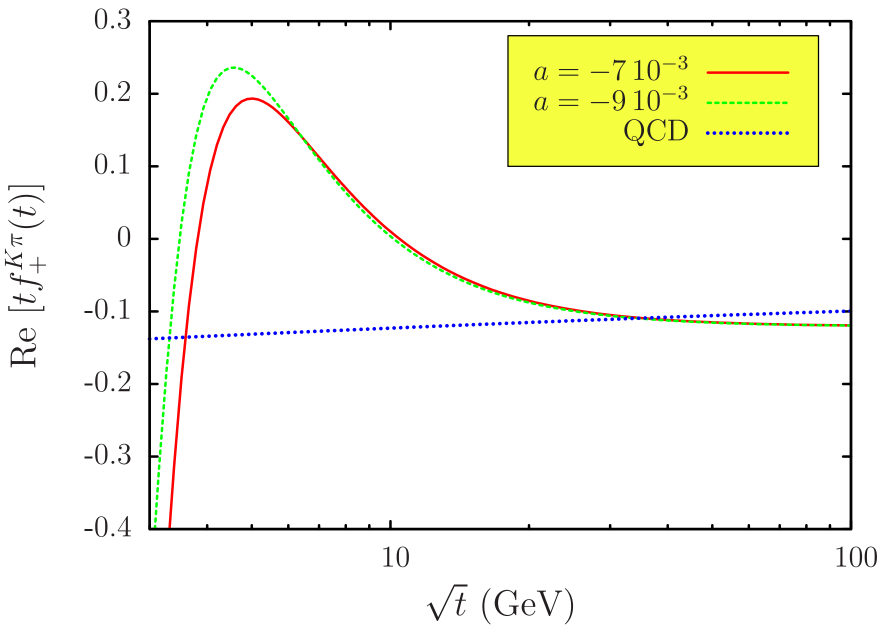
$$R_{PDG} = (13.5 \pm 0.05)\, 10^{-3} \qquad a \simeq -7.\, 10^{-3}$$

$$\left.\begin{aligned} R_{Babar} &= (12.48 \pm 0.009 \pm 0.054)\, 10^{-3} \\ R_{Belle} &= (12.12 \pm 0.006 \pm 0.039)\, 10^{-3} \end{aligned}\right\} \quad a \simeq -9.\, 10^{-3}$$

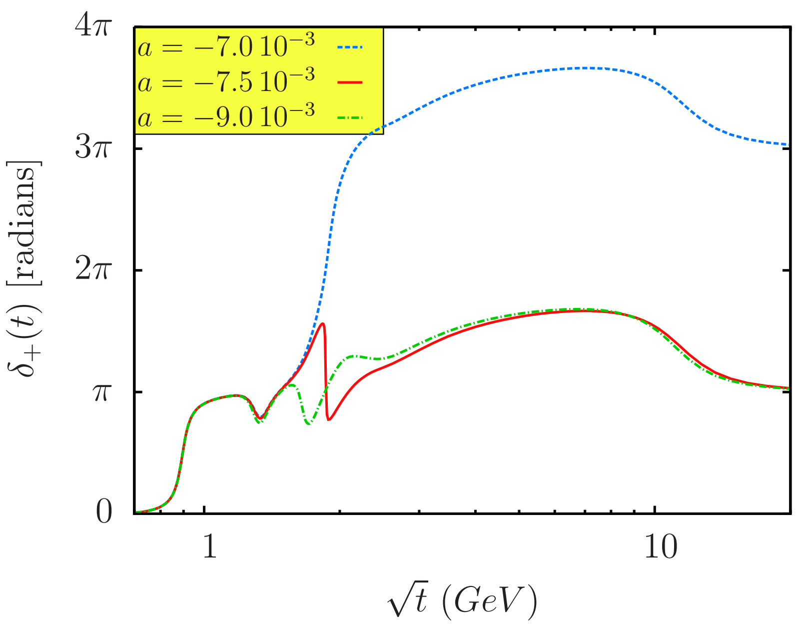
Asymptotic behaviour

* Sum rule imposed: $\int_{(m_K+m_\pi)^2}^{\infty} dt' \mathrm{Im}\, f_+^{K\pi}(t') = 16\pi^2\sqrt{2}\alpha_s$
  with $\alpha_s = 0.2$

* Form factor



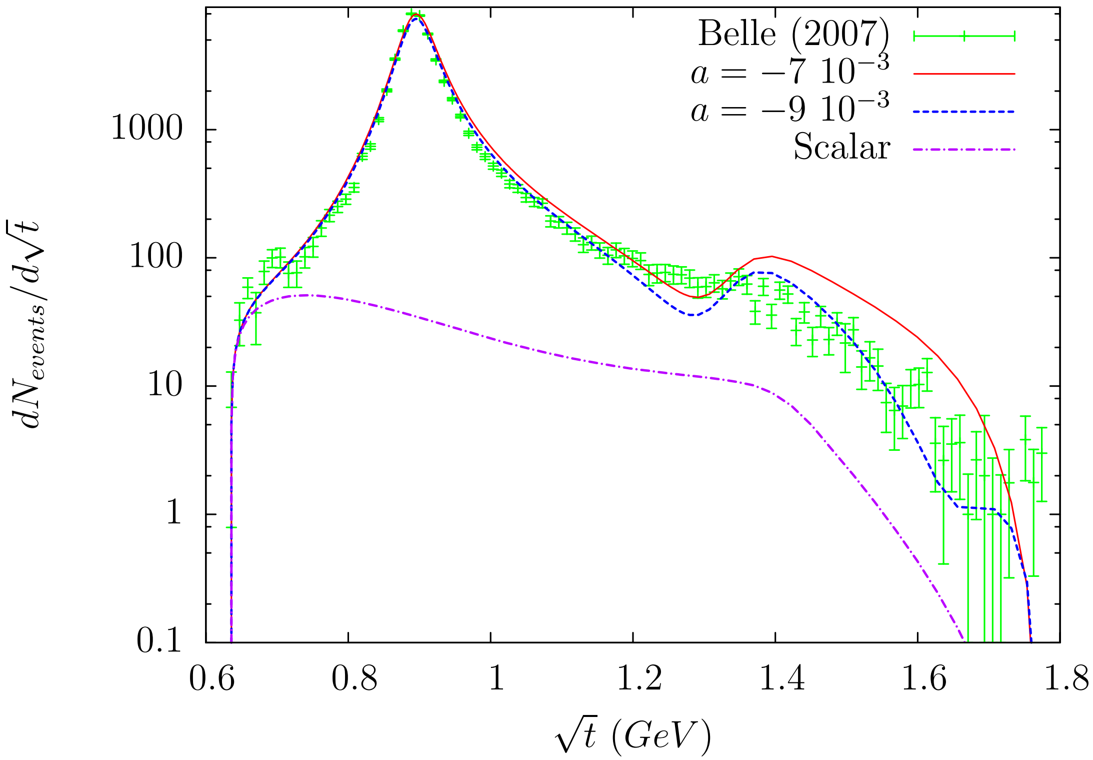
* Phase behaviour



Comparison with τ decay data

* Recently Belle (2007): $\tau \to K_S^0 \pi^- \nu_\tau$ and Babar (2007): $\tau \to K^- \pi^0 \nu_\tau$ [statistics: $\times 10^3$ compared to Aleph (1999)]

* Scalar form factor $f_0^{K\pi}(t)$ also needed [similar method, Jamin, Oller, Pich (2001)]



$*$ Comments:

– $K^*(892)$ isospin breaking
$m_{K^{*0}} \approx m_{K^{*+}} \approx 895$ MeV, $\Gamma_{K^{*0}} - \Gamma_{K^{*+}} \approx 6$ MeV

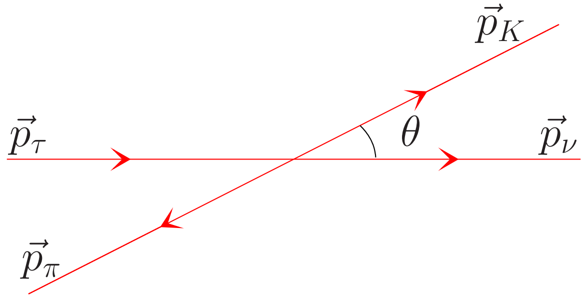
– $K^*(1410)$: more pronounced structure in LASS

$*$ Low energy expansion:

$$f_+^{K\pi}(t) = f_+^{K\pi}(0)\,[1 + \lambda'_+\, t/m_{\pi^+}^2 + 1/2\lambda''_+\, t^2/m_{\pi^+}^4 + ...]$$

	λ'_+	λ''_+
$a = -9\,10^{-3}$	$25.5\,10^{-3}$	$1.25\,10^{-3}$
Kloe(2006,2007)	$(25.6 \pm 1.5 \pm 0.9)\,10^{-3}$	$(1.4 \pm 0.7 \pm 0.4)\,10^{-3}$

Separation of $f_0^{K\pi}$ and $f_+^{K\pi}$ in τ decay



∗ Double differential decay rate:

$$\frac{d^2\Gamma}{d\sqrt{t}\, d\cos\theta} = \frac{3V_{us}^2 G_F^2}{256\pi^3}\left(\frac{m_\tau^2}{t} - 1\right)^2 \frac{q_{K\pi}(t)}{m_\tau}$$

$$\Big\{ |f_+^{K\pi}(t)|^2 \lambda_{K\pi}(t) \left[\left(1 - \frac{t}{m_\tau^2}\right)\cos^2\theta + \frac{t}{m_\tau^2}\right]$$

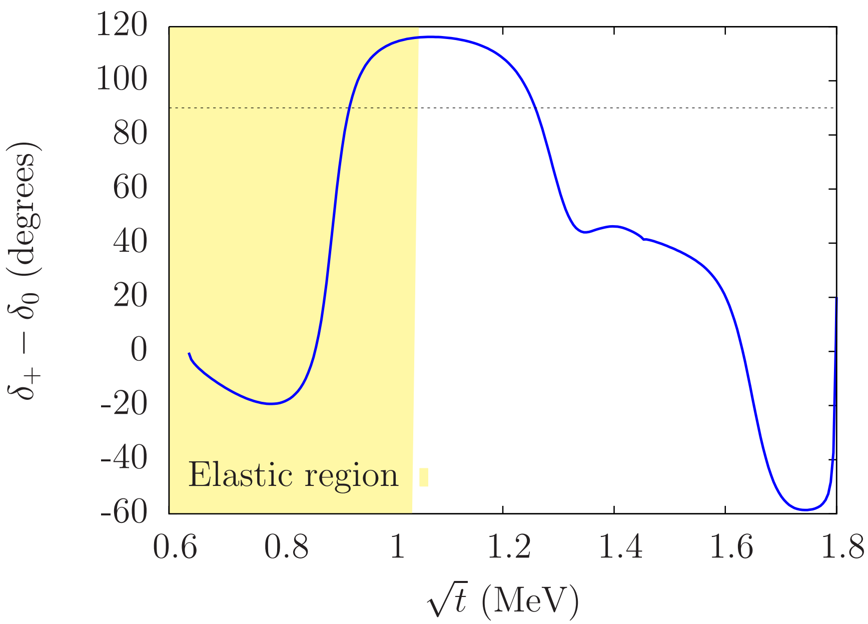
$$+ |f_0^{K\pi}(t)|^2 (m_K^2 - m_\pi^2)^2$$

$$+ 2|f_+^{K\pi}(t) f_0^{*K\pi}(t)| \cos(\delta_+ - \delta_0)(m_K^2 - m_\pi^2)\lambda_{K\pi}^{\frac{1}{2}}(t)\cos\theta \Big\}$$

with:

$$\lambda_{K\pi}(t) = (t - (m_K + m_\pi)^2)(t - (m_K - m_\pi)^2)$$

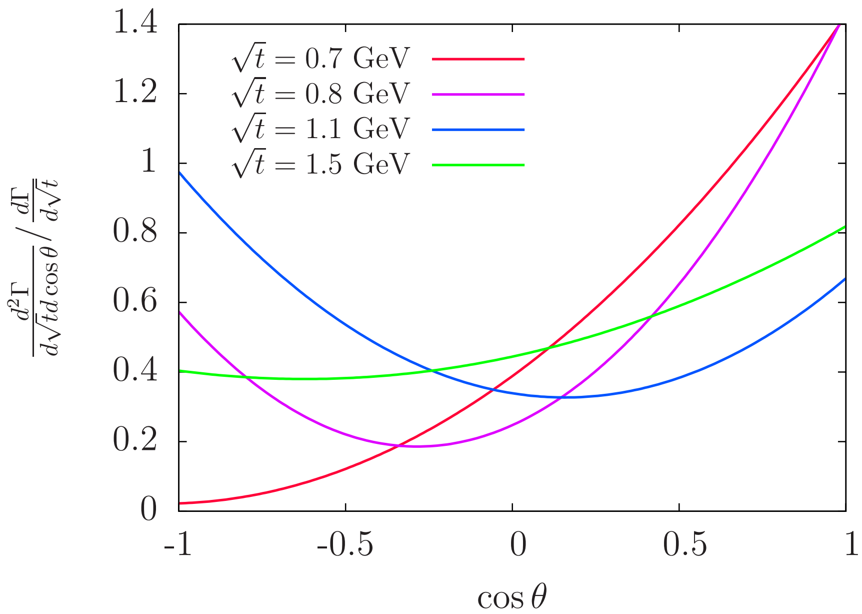
* Relative phase of form factors



* Angular distributions

– Asymmetry visible

– Forward-backward asymetry changes with E



Conclusions

$\ast$ Pseudo-goldstone interactions are important probes of effective theory $[$ $\pi\pi$, πK, $\pi\eta$?,... $]$

$\ast$ (Precise) experimental info useful (not necessarily low energy) combined with general theoretical constraints

$\ast$ More experimental info to come on πK: πK atom, τ decays, D semi leptonic decays

$\ast$ $O(p^6)$ couplings: resonance saturation method unreliable. Use global fits, lattice QCD (scattering lengths, correlation functions...)