$1+1$ Dimensional Large-N QCD and Baryons

Govind S. Krishnaswami

Durham University, UK.

Strong Frontiers, Bangalore, 16 January, 2009.

Based on

1. S. G. Rajeev, *Quantum Hadron Dynamics in 2 Dimensions*, Int.J.Mod.Phys.A9:5583, 1994.

2. G. S. Krishnaswami and S. G. Rajeev, *A Model of Interacting Partons for Hadronic Structure Functions*, Physics Letters B, 441 (1998) 429-436.

3. V. John, G. S. Krishnaswami and S. G. Rajeev, *Parton Model from bi-local solitonic picture of baryon in two dimensions*, Phys. Lett. B 492, 63 (2000).

4. V. John, G. S. Krishnaswami and S. G. Rajeev, *An interacting parton model for quark and anti-quark distributions in the baryon*, Phys. Lett. B 487, 125 (2000).

5. G. S. Krishnaswami, *Large-N Limit as a Classical Limit: Baryon Two-dimensional QCD and Multi-Matrix Models*, PhD Thesis University of Rochester 2004 [arXiv:hep-th/0409279].

Abstract

This talk is based on joint work with S. G. Rajeev and V. John. We will explain how the multi-color limit of $1+1$ QCD can be formulated as a classical hamiltonian system by analogy with the Hartree approximation of atomic physics. The basic dynamical variable is a gauge-invariant quark bilinear. 't Hooft's meson spectrum is recoved via small oscillations around the vacuum. The baryon arises as a topological soliton and its form factor is determined variationally. This is not a low energy effective model but is valid at all energies. It provides an approximation to the non-perturbative proton structure function measured in deep inelastic scattering.

Quantum Hadron Dynamics

• A challenging problem of theoretical physics is to find a reformulation of QCD in terms of gauge invariant variables, a Quantum Hadron Dynamics.

• There are indications that such a reformulation exists, and that it is a sort of string field theory or a theory of Wilson loop variables.

• In 't Hooft's multi-colour limit we see glimpses of such a possibility since mesons and glueballs are weakly interacting if we consider planar Feynman diagrams alone.

• More precisely, the large N limit of QCD is a classical theory for gauge invariant observables $\frac{\mathrm{tr}}{N}M$. Their fluctuations become small.

$$\langle \frac{\mathrm{tr}}{N} M_1 \ \frac{\mathrm{tr}}{N} M_2 \rangle = \langle \frac{\mathrm{tr}}{N} M_1 \rangle \ \langle \frac{\mathrm{tr}}{N} M_2 \rangle + \mathcal{O}(\frac{1}{N})$$

Strategy:

- Reformulate $N = \infty$ QCD as a classical theory of gauge invariant variables.

- Develop methods to solve the classical dynamics and determine the hadron structure functions and spectrum.

- Recover the $N = 3$ QCD as the quantization of this theory in $\frac{1}{N}$.

- These ideas have largely been realized in $1 + 1$ dimensions.

- Does such an approach have a **precedent in physics** where it provides a good approximation?

- It is instructive to see how a similar but simpler situation can be dealt with in a more mature branch of physics such as **atomic physics**.

The Ground state of the Atom

- Basic problem of atomic physics: find ground state of m-electron atom.

$$\begin{aligned} H &= \sum_{a=1}^{m} \Big[-\nabla_a^2 + V(x_a) \Big] + \sum_{a<b} G(x_a - x_b) \\ V(x_a) &= -\frac{Ze^2}{|x_a|}; \quad G(x_a - x_b) = \frac{e^2}{|x_a - x_b|} \\ E_0 &= \min_{||\psi||=1} < \psi | H | \psi > \end{aligned}$$

- $\psi(x_1, \cdots, x_m) \rightarrow$ totally anti-symmetric (Pauli principle).

- If $||\psi|| = 1$, we have $\langle \psi | H | \psi \rangle = \text{ tr } h\rho_1 + \text{ tr } G_2\rho_2, \quad h = -\nabla^2 + V$

- $\rho_1(x, y)$ and $\rho_2(x, y; z, u) \rightarrow$ 1 and 2 particle density matrices.

$$G_2(x, y; z, u) = \frac{1}{2}\Big[\delta(x - z)\delta(y - u) - \delta(x - u)\delta(y - z)\Big]G(x - y)$$

- The problem of finding the exact ground state is **difficult** for $m > 1$.

One and two particle density matrices

$$\rho_1(x,y) = m \int \psi^*(x, x_2, \cdots, x_m)\; \psi(y, x_2, \cdots, x_m)\; dx_2 \cdots dx_m$$

$$\rho_2(x,y;z,u) = \binom{m}{2} \int \psi^*(x, y, x_3, \cdots, x_m)\; \psi(z, u, x_3, \cdots, x_m)\; dx_3 \cdots dx_m$$

- m = number of electrons

1st Quantized Hartree-Fock Theory

- Basic idea of Hartree-Fock is to use the Slater determinant variational ansatz

$$\psi(x_1, \cdots x_m) = \frac{1}{\sqrt{m!}} \det \begin{pmatrix} u_1(x_1) & \cdots & u_1(x_m) \\ \vdots & \ddots & \vdots \\ u_m(x_1) & \cdots & u_m(x_m) \end{pmatrix}; \qquad (u_a, u_b) = \delta_{a,b}$$

$$E_{\mathrm{HF}}(u) = < \psi | H | \psi >$$

- Slater ansatz $\rightarrow$ good variational approximation to the ground state.

- But this ansatz has a redundancy $\quad u_a(x) \mapsto g_a^b \, u_b(x) \Rightarrow \psi \mapsto \det g \;\; \psi$

- $g \in U(m) \Rightarrow det(g)$ is a phase $\Rightarrow$ physical state does not change.

Atomic Hartree-Fock in terms of Projectors

- Spurious degrees of freedom make $u_a(x)$ inconvenient as variational quantities.

- The number of redundant degrees of freedom grows with m.

- There is a more economical way to formulate Hartree-Fock theory.

- $P \longrightarrow$ projection operator to subspace spanned by $\{u_a\}$

$$P(x,y) = \sum_{a=1}^{m} u_a^*(x)u_a(y) \ ; \quad P^2 = P, \qquad P^\dagger = P$$

- $P \Rightarrow$ electron density matrix. $P^2 = P \Rightarrow$ Pauli Exclusion Principle.

- Then the 1 and 2 particle density matrices take a simple form

$$\rho_1 = P; \qquad \rho_2(x,y;z,u) = P \wedge P = \frac{1}{2}\Big[P(x,z)P(y,u) - P(x,u)P(y,z)\Big]$$

Grassmannians and Atomic Hartree-Fock

- The Hartree-Fock energy is

$$E_{\rm HF}(P) = \int dx\Big[\Big(\frac{\partial^2 P(x,y)}{\partial x \partial y}\Big)_{x=y} + V(x)P(x,x)\Big] + \int dxdy G(x-y)\Big[P(x,x)P(y,y) - P(x,y)P(y,x)\Big]$$

- So static Hartree-Fock-Slater theory can be summarized as

$$E^0_{\rm HF} = \min_{\rm Gr_m} E_{\rm HF}(P)$$

- Gr_m is the ∞-dim Grassmannian manifold of rank m projection operators.

$$Gr_m = \{P|P^\dagger = P, P^2 = P,\ \mathrm{tr}\ P = m\} = U(\mathcal{H})/(U(m)\times U(\mathcal{H}))$$

- Gr $\longrightarrow$ curved manifold, connected components labeled by rank of projection, (number of electrons)

$$\mathrm{Gr} = \bigcup_{m\geq 0} \mathrm{Gr}_m$$

Hartree Fock as a Classical Limit

- As in any mean field theory, quantum fluctuations are small in Hartree-Fock theory.
- Is there a classical dynamical system which is equivalent to Hartree-Fock theory? YES!
- Can we recover atomic physics by quantizing this classical system? YES!
- The classical limit corresponding to Hartree theory could not be $\hbar \to 0$ (atom unstable).
- Hartree-Fock corresponds to hamiltonian dynamics on the Grassmannian.
- Quantization of this classical system in $1/N$ holding $\hbar = 1$ fixed gives back full atomic theory.
- $N =$ number of replicas or 'colours' of each electron.
- $N = 1$ in nature (standard atomic physics); $N \to \infty$ classical limit is Hartree Theory

Second Quantized Hartree-Fock

- To see this it is convenient to work in a second quantized language.

- Atomic Hamiltonian

$$\hat{H} = \int dx \; a^{\dagger}(x) \; h \; a(x) + \int dx \; dy \; G(x-y) \; a^{\dagger}(x) \; a^{\dagger}(y) \; a(y) \; a(x)$$

- Creation-annihilation operators satisfy canonical anti commutation relations (CAR).

$$[a^{\dagger}(x), a(y)]_{+} = \delta(x-y); \quad [a^{\dagger}(x), a^{\dagger}(y)]_{+} = 0; \quad [a(x), a(y)]_{+} = 0$$

- CAR algebra has a representation on the fermionic Fock space

$$|\psi\rangle \quad \in \quad \oplus_{m=0}^{\infty} \Lambda^{m}(L^{2}(R^{3} \otimes C^{2}_{spin}))$$

Second Quantized Hartree-Fock

- Introduce a **fictitious 'colour'** quantum number for the electrons $\alpha = 1, \cdots, N$.

- N **'replicas'** of each of the m electrons.

$$[a^{\dagger\alpha}(x), a_\beta(y)]_+ = \delta^\alpha_\beta \delta(x-y); \quad [a^{\dagger\alpha}(x), a^{\dagger\beta}(y)]_+ = 0; \quad [a_\alpha(x), a_\beta(y)]_+ = 0$$

- This CAR algebra also has a representation on a coloured Fermionic Fock space $\mathcal{F}$.

- But require physical states $\mathcal{F}_0 \subset \mathcal{F}$ to be annihilated by colour charge

$$Q^\alpha_\beta = \int dx [a^{\dagger\alpha}(x) a_\beta(x) - \frac{1}{N} \delta^\alpha_\beta a^{\dagger\gamma}(x) a_\gamma(x)]$$

$$Q^\alpha_\beta |\psi> = 0 \Rightarrow |\psi> \in \mathcal{F}_0$$

Coloured Electrons

- The replicated Hamiltonian is

$$\hat{H} = \frac{1}{N}\int dx \;\; a^{\dagger\alpha}(x)\; h\; a_\alpha(x)$$

$$+ \frac{1}{2N}\int dxdy\; G(x-y)\Big[a^{\dagger\alpha}(x)a^{\dagger\beta}(y)a_\beta(y)a_\alpha(x) + a^{\dagger\alpha}(x)a^{\dagger\beta}(y)a_\alpha(y)a_\beta(x)\Big]$$

- For $N = 1$, this reduces to the **standard atomic Hamiltonian** and $Q^\alpha_\beta = 0$.

- We will see for $N \to \infty$, this becomes **classical dynamics on the Grassmannian**, which is the same as **Hartree-Fock Theory**.

Coloured Electrons

• To see this, introduce the **colour-singlet bilocal observables** on the $m-$electron physical Fock space $\mathcal{F}_{0m}$.

$$\hat{P}(x,y) = \frac{1}{N}\hat{a}^{\dagger\alpha}(x)\hat{a}_{\alpha}(y)$$

• $\hat{P}$ satisfy a quadratic constraint.

• If $|\psi\rangle,\ |\psi'\rangle\ \in \mathcal{F}_{0m}$

$$< \psi'| \int : \hat{P}(x,y)\hat{P}(y,u) : dy|\psi >= \left(1 - \frac{m}{N}\right) < \psi'|\hat{P}(x,u)|\psi >$$

Quantum Theory of Coloured Electrons

- We can write the Hamiltonian in terms of $\hat{P}$

$$\hat{H} = \int dx\Big[\Big(\frac{\partial^2 \hat{P}(x,y)}{\partial x \partial y}\Big)_{x=y} + V(x)\hat{P}(x,x)\Big] + \int dxdyG(x-y)\Big[: \hat{P}(x,x)\hat{P}(y,y) : - : \hat{P}(x,y)\hat{P}(y,x) : \Big]$$

- $\hat{P}$ satisfy commutation relations of **infinite dimensional Unitary Lie Algebra**

$$[\hat{P}(x,y), \hat{P}(z,u)] \;=\; \frac{1}{N}\Big(\delta(y-z)\hat{P}(x,u) - \delta(x-u)\hat{P}(z,y)\Big)$$

$$\mathrm{tr}\ \hat{P} \;=\; m = \text{ number of electrons}$$

- $\frac{1}{N} \rightarrow$ uncertainty in a simultaneous measurement of different components of $\hat{P}(x,y)$.

Classical Limit of Colored Electrons

- $N \to \infty$: **classical limit** whose **phase space** is the **Grassmannian**, due to the quadratic constraint $< \psi' | \int : \hat{P}(x,y)\hat{P}(y,u) : dy | \psi >= \left(1 - \frac{m}{N}\right) < \psi' | \hat{P}(x,u) | \psi >$

- Slater-type states $\to$ coherent states, minimum uncertainty states, have a good classical limit.

- Time dependent Hartree-Fock is hamiltonian dynamics on the Grassmannian.

$$\frac{\partial P(x,y)}{\partial t} = \{E_{\mathrm{HF}}(P), P(x,y)\}$$

- Success of Hartree-Fock shows: $N \to \infty$ a good approximation even for $N = 1$.

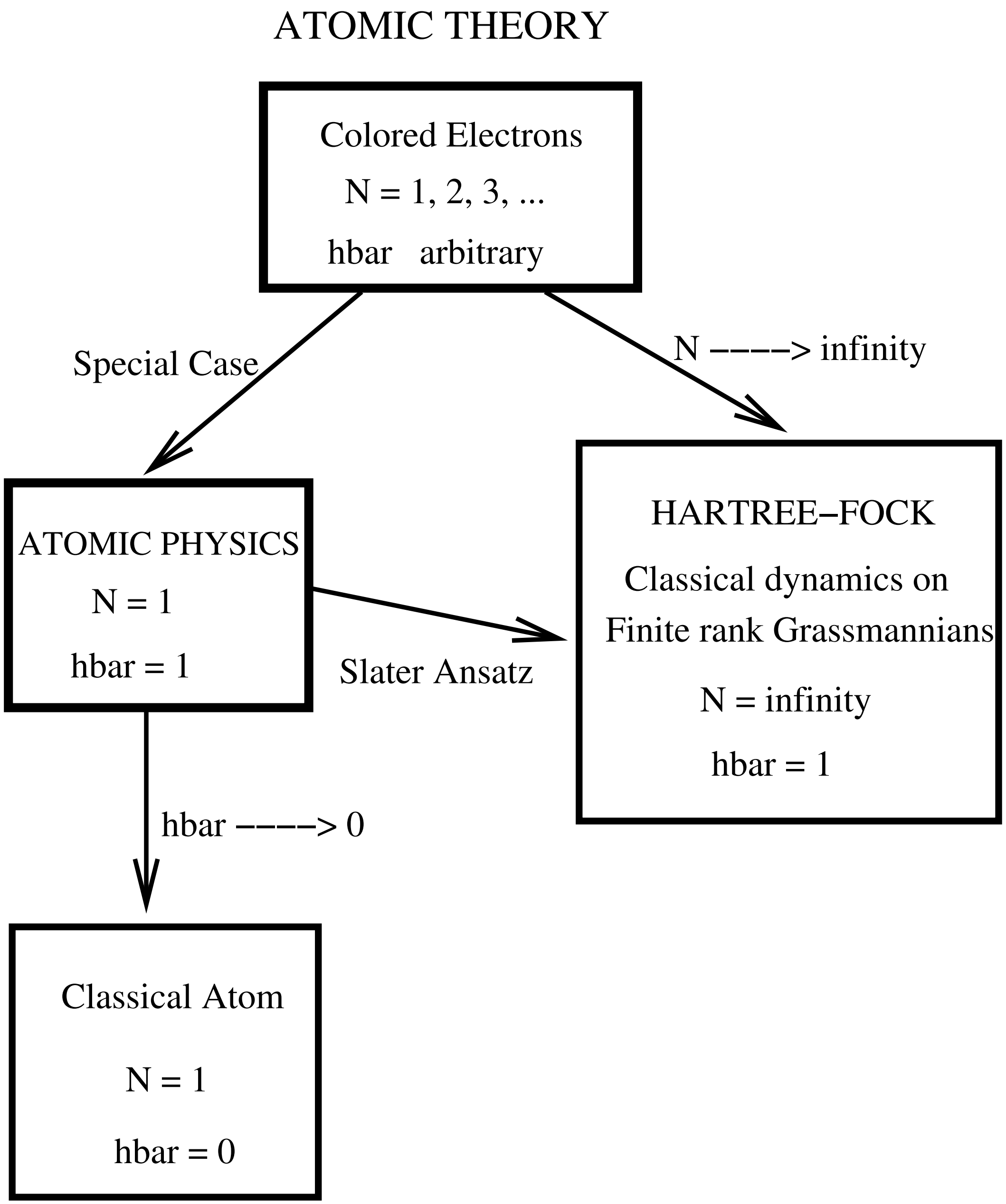
ATOMIC THEORY
Colored Electrons
N = 1, 2, 3, ...
hbar arbitrary
Special Case
N ----> infinity
ATOMIC PHYSICS
N = 1
hbar = 1
Slater Ansatz
HARTREE–FOCK
Classical dynamics on
Finite rank Grassmannians
N = infinity
hbar = 1
hbar ----> 0
Classical Atom
N = 1
hbar = 0

Figure 1:

$1+1$ dimensional QCD

- 2d QCD is a finite QFT. Coupling constant g has dimensions of mass.

$$S_{QCD} = \frac{N}{4g^2}\int \text{ tr } F_{\mu\nu}F^{\mu\nu}d^2x + \int \bar{q}\,[i\gamma\cdot D - m]\,qd^2x.$$

- $N=3$ is number of colors. m is current quark mass, $\frac{m}{g}\to 0$ is chiral limit.

- Null coordinates: $x = x^1$, $t = x^0 - x^1$; $\frac{\partial}{\partial t}$ is time like and $\frac{\partial}{\partial x}$ is null.

$$ds^2 = (dx^0)^2 - (dx^1)^2 = dt^2 + 2dxdt \;\Rightarrow\; \eta^{\mu\nu} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \eta_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

- Initial value surface is $t=0$

- Energy is $E = p_t = p_0 = -i\partial_t$ and *null* momentum is $p_x = p_0 + p_1 = -i\partial_x$

Eliminating Gluons

- Work in null gauge $A_x = A_0 + A_1 = 0$, with $q = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta \\ \chi \end{pmatrix}$ and $A_t = A_0$.

$$S_{QCD} = \frac{N}{2g^2} \int \text{ tr } (\partial_x A_t)^2 dxdt + \int dxdt [\chi^\dagger(-i\partial_t - A_t)\chi + \frac{1}{2}(\eta^\dagger(-i\partial_x)\eta - \chi^\dagger(-i\partial_x)\chi) - \frac{m}{2}(\chi^\dagger\eta + \eta^\dagger\chi)]$$

- Eliminate $\eta = \frac{m}{p}\chi$ and $A_t = -\frac{g^2}{N} \int dy \; \chi^{\dagger a}(y)\chi_b(y) \; \frac{1}{2}|x-y|$.

$$\begin{aligned} S_{QCD} \;=\; & \int dtdx \; \chi^\dagger \; (-i\partial_t) \; \chi - \int dtdx \; \chi^\dagger \; \frac{1}{2}(p + \frac{m^2}{p}) \; \chi \\ & - \frac{g^2}{2N} \int dtdxdy \;\; \text{tr } \chi^\dagger(y)\chi(y) \; \frac{|x-y|}{2} \; \chi^\dagger(x)\chi(x) \end{aligned}$$

- Gluons eliminated: quarks χ interact via linear potential.

$1 + 1$ dimensional QCD

• Perturbatively, in the $N \to \infty$ limit holding $\hbar$ and $\tilde{g}^2 = g^2 N$ fixed, planar diagrams dominate ('t Hooft).

• 't Hooft found meson spectrum by summing planar diagrams and obtaining a **linear integral equation for their masses**.

• NOT the whole story even in the large N limit: Where are BARYONS?.

• Early proposal of SKYRME: baryons arise as solitons of non-linear theory of mesons. This was realized in the low energy effective models but not in QCD.

• Witten suggested that baryons may be described by a sort of Hartree-Fock approximation in the large N limit of QCD, but the idea was not implemented.

$1 + 1$ dimensional large-N QCD

- In $N \to \infty$ limit, **fluctuations in gauge invariant observables are small**.

$$\langle M_1 M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle + \mathcal{O}(1/N)$$

- $N \to \infty$ is a classical limit different from classical Chromodynamics.

- Reformulate $N \to \infty$ 2d QCD as a classical theory of colour singlet quark bilinears.

- Allows us to understand baryons in 2d, which was an outstanding challenge.

Quantum Hadron Dynamics in 2 Dimensions

- Introduce a bilocal colour singlet meson variable: a bosonization

$$\hat{M}(z,w) = -\frac{2}{N} : \chi^{\dagger a}(z) P \exp^{\int_z^w A_x dx} \chi_a(w) : \quad \rightarrow M(z,w) \quad \text{as} \quad N \rightarrow \infty$$

- z, w are null separated and $P \exp^{\int_z^w A_x dx} = 1$ in $A_x = A_0 + A_1 = 0$ gauge.

- Classical **meson variable** $M(z,w) \rightarrow$ **Master field**.

- Anticommutators of $\chi, \chi^\dagger$, $\Rightarrow$ Poisson algebra of $M(x,y)$. It is a central extension of an ∞-d unitary Lie algebra. ($\epsilon(x,y) = \int [dk] e^{ik(x-y)} \text{ sgn } (k) = \frac{i}{2\pi} \mathcal{P}(\frac{1}{x-y})$)

$$\{M(x,y), M(z,u)\} \;=\; -i[\delta(y-z)(\epsilon(x,u) + M(x,u)) - \delta(x-u)(\epsilon(z,y) + M(z,y))]$$

- $M(x,y)$ must be constrained: same colourless degrees of freedom as $\chi(x)$.

Dirac Vacuum and general states

- Kinetic energy and null-momentum have the same sign.

$$m^2 = p_o^2 - p_1^2 = (2p_t - p_x)p_x \quad \Rightarrow \quad E = p_t = \frac{1}{2}(p + \frac{m^2}{p})$$

- One particle Hilbert space is a sum of $p < 0$ and $p > 0$ states.

$$\mathcal{H} = L^2(R) = \mathcal{H}_- \oplus \mathcal{H}_+$$

- Dirac vacuum: fill negative momentum states (-1), empty positive ones $(+1)$

$$\epsilon = \text{ sgn } (p) = \begin{pmatrix} \ddots & & & & & \\ & -1 & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & \ddots \end{pmatrix}, \quad \epsilon^\dagger = \epsilon, \quad \epsilon^2 = 1;$$

- General state: $\Phi^\dagger = \Phi, \Phi^2 = 1$ with $\Phi - \epsilon$ 'finite', better work with $M = \Phi - \epsilon$.

Phase space of classical hadron theory

- **Phase space** is a curved manifold, an infinite dimensional **Grassmannian** of projection operators which deviate by a finite rank operator from the Dirac vacuum ϵ.

$$Gr(\mathcal{H}, \epsilon) = \{M^\dagger = M, (\epsilon + M)^2 = 1, \ \mathrm{tr}\ M^\dagger M < \infty\}$$

- $P = -\frac{1}{2}(M + \epsilon - 1)$ satisfies $P^2 = P$, quark density matrix is a projection.

- $M \longrightarrow$ normal ordered version of P and $\Phi = \epsilon + M$

- Grassmannian is the coadjoint orbit of the point ϵ by the restricted unitary group

$$U(\mathcal{H}, \epsilon) = \{U \mid U^\dagger U \ = \ UU^\dagger = 1, \quad \mathrm{tr}\, [\epsilon, U]^\dagger\, [\epsilon, U] \ < \ \infty\}$$

- Carries unitary group action $M \mapsto UMU^\dagger + U\epsilon U^\dagger - \epsilon$

Hamiltonian of classical hadron theory

- Starting from action of 2d QCD, we can express the hamiltonian in terms of $M(x,y)$.

- The Hamiltonian is a quadratic function on the Grassmannian

$$E(M) = -\frac{1}{4}\int (p+\frac{\mu^2}{p})\tilde{M}(p,p)[dp] + \frac{\tilde{g}^2}{8}\int M(x,y)M(y,x)\frac{|x-y|}{2}dxdy$$

- $\tilde{M}(p,q) = \int dx\ dy\ e^{i(px-qy)}\ M(x,y)$

- $\mu^2 = m^2 - \frac{\tilde{g}^2}{\pi}$: finite renormalization of current quark mass. Due to reordering of operators.

Hamilton's equations of motion

• Hamilton's equations are quadratically **non-linear integral equations** for $M(x, y)$

$$\frac{\partial M}{\partial t} = \{E(M), M\} = -i[E'(M), \epsilon + M]$$

$$\begin{aligned} \frac{i}{2}\frac{\partial M(x,y)}{\partial t} = & \int dz[h(x-z)M(z,y) - M(x,z)h(z,y)] \\ & +\tilde{g}^2 \int dz[G(y-z)\epsilon(x,z)M(z,y) - G(z-x)\epsilon(z,y)M(x,z)] \\ & +\tilde{g}^2 \int dz \; M(x,z)M(z,y)[G(y-z) - G(z-x)] \end{aligned}$$

• $G(x-y) = -\frac{1}{2}|x-y|$ and $h = \frac{1}{2}(p + \frac{\mu^2}{p})$ are integral kernels of potential energy and kinetic energy.

Meson Spectrum

- Simplest static solution is $M = 0$ or $\Phi = \epsilon$, Dirac vacuum.

- Linearization of Hamilton's equations about $M = 0$ describe small oscillations

- Satisfying 't Hooft's linear integral equation for meson spectrum

$$\tilde{M}(p,q,t) \;=\; e^{i\omega t}\chi(\xi), \quad \xi = p/P \quad P = p - q$$

$$\mathcal{M}^2\chi(\xi) \;=\; [\frac{\mu^2}{\xi} + \frac{\mu^2}{(1-\xi)}]\chi(\xi) - \frac{\tilde{g}^2}{\pi}\mathcal{P}\int_0^1 \frac{\chi(\eta)}{(\xi-\eta)^2}d\eta$$

- $\mathcal{M}^2 = 2\omega P - P^2$ is the square of the meson mass and P is the total momentum.

- What about baryons?

Baryons

- Grassmannian phase space is a **disconnected manifold**.

- Connected components labeled by a **topological invariant**, **virtual rank** of the projection operator, physically the **baryon number**.

$$B = -\frac{1}{2}\,\mathrm{tr}\, M = \frac{1}{N}\int dx :\chi^a(x)\chi_a(x):$$

- Extrema of energy on $B = 1$ sector of phase space describe baryons.

- Higher baryon number sectors describe $1 + 1$d nuclei.

- Classical mechanics on an infinite dimensional curved phase space is difficult.

- But it is simpler than the QFT we started out with.

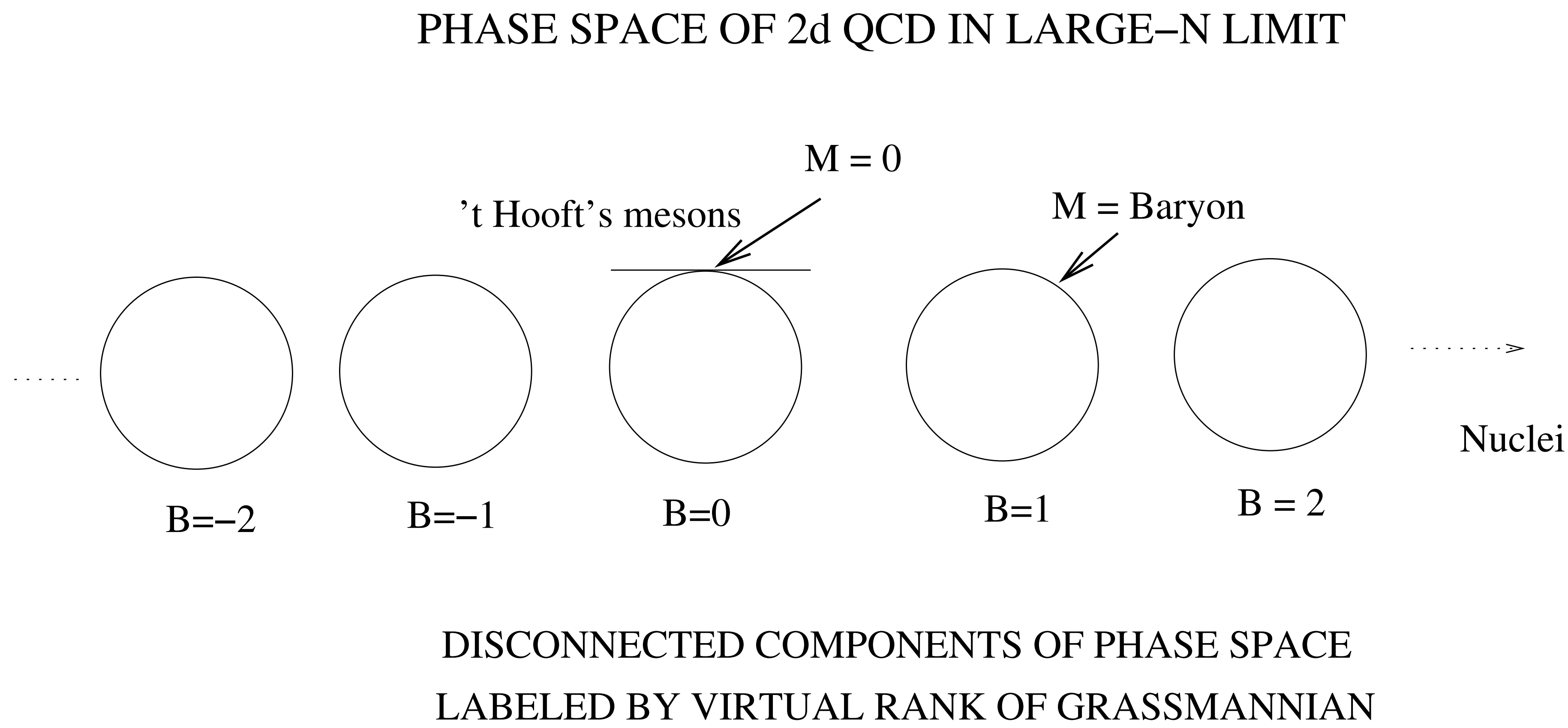
PHASE SPACE OF 2d QCD IN LARGE−N LIMIT
M = 0
’t Hooft’s mesons
M = Baryon
Nuclei
B=−2
B=−1
B=0
B=1
B = 2
DISCONNECTED COMPONENTS OF PHASE SPACE
LABELED BY VIRTUAL RANK OF GRASSMANNIAN

Figure 2:

Submanifolds of Phase Space

• Structure of baryon $\rightarrow$ approximately determine by **restricting dynamics to finite dimensional sub-manifolds** of the $B = 1$ component of phase space.

• e.g. The rank 1 submanifold corresponds to the **separable ansatz**

$$\tilde{M}(p,q) = -2\tilde{\psi}(p)\tilde{\psi}^*(q), \quad ||\psi|| = 1, \quad \tilde{\psi}(p) = 0 \text{ if } p < 0 \quad (\epsilon\psi = \psi)$$

• Automatically satisfies constraints $(M+\epsilon)^2 = 1$ and $-\frac{1}{2}\,\mathrm{tr}\, M = B = 1$.

• Dynamics on the rank 1 submanifold defines a self-contained Hamiltonian system.

$$\{\tilde{\psi}(p), \tilde{\psi}(p')\} \ = 0 = \ \{\tilde{\psi}^*(p), \tilde{\psi}^*(p')\}, \{\tilde{\psi}(p), \tilde{\psi}^*(p')\} = -i\ 2\pi\delta(p-q)$$

$$E(\psi) = \int \ \frac{1}{2}\,[p + \frac{\mu^2}{p}]\ |\tilde{\psi}(p)|^2[dp] + \ \frac{1}{2}\,\tilde{g}^2\ \int |\psi(x)|^2|\psi(y)|^2\frac{|x-y|}{2}dxdy.$$

Quantization of reduced system on rank-1 submanifold

- Reduced dynamical system on rank-1 submanifold
- It is the Hartree approximation to a **relativistic bound state problem**
- N-quarks interacting via a **linear potential** in the large N limit.
- To see this, quantize this reduced dynamical system, i.e. go back to finite N.
- Constraint $||\psi||^2 = 1$ implemented by restricting to states $|V\rangle$ annihilated by it

$$\int dx \;\; \hat{\psi}^*(x)\, \hat{\psi}(x)\, |V\rangle \;=\; |V\rangle.$$

Interacting Bosons

• Poisson brackets $\longrightarrow$ commutation relations, $\frac{1}{N}$ plays role of $\hbar$.

$$[\tilde{\hat{\psi}}(p),\tilde{\hat{\psi}}(p')]=0=[\tilde{\hat{\psi}}^{\dagger}(p),\tilde{\hat{\psi}}^{\dagger}(p')],\ [\tilde{\hat{\psi}}(p),\tilde{\hat{\psi}}^{\dagger}(p')]=\frac{1}{N}2\pi\delta(p-p').$$

• Representation for these commutation relations is given by the canonical commutation relations of bosonic creation-annihilation operators:

$$[\tilde{\hat{b}}(p),\tilde{\hat{b}}(p')]=0=[\tilde{\hat{b}}^{\dagger}(p),\tilde{\hat{b}}^{\dagger}(p')],\quad [\tilde{\hat{b}}(p),\tilde{\hat{b}}^{\dagger}(p')]=2\pi\delta(p-q).$$

• $\hat{\psi}=\frac{1}{\sqrt{N}}\hat{b},\quad \hat{\psi}^{\dagger}=\frac{1}{\sqrt{N}}\hat{b}^{\dagger}.$

• The constraint becomes

$$< V|\int_0^{\infty}\hat{b}^{\dagger}(p)\hat{b}(p)[dp]|V>=N$$

• Thus we have a system of N **bosons**, so N must be an integer.

Interacting Bosons

$$N\hat{E}_1 = \int_0^\infty \frac{1}{2}[p + \frac{\mu^2}{p}]\tilde{b}^\dagger(p)\tilde{b}(p)\ [dp] + \frac{\tilde{g}^2}{2N}\int b^\dagger(x)b^\dagger(y)\frac{|x-y|}{2}b(y)b(x)\ dxdy.$$

- In the momentum basis, an N particle momentum eigenstate is

$$|p_1, p_2, \cdots, p_N\rangle = b^\dagger(p_1)b^\dagger(p_2)\cdots b^\dagger(p_N)|0\rangle.$$

- A general state $|\psi >$ containing N particles is

$$|\psi >= \int [dp_1]\cdots[dp_N]\ \tilde{\psi}(p_1,\cdots,p_N)\ |p_1,\cdots,p_N\rangle.$$

- The expectation value of the hamiltonian in such a state is

$$\begin{aligned} < \psi|N\hat{E}|\psi > \ =& \ \int_0^\infty \sum_{i=1}^N \frac{1}{2}[p_i + \frac{\mu^2}{p_i}]|\tilde{\psi}(p_1,\cdots,p_N)|^2[dp_1]\cdots[dp_N] \\ &+ \frac{\tilde{g}^2}{2N}\int_0^\infty \sum_{i\neq j}\frac{|x_i - x_j|}{2}|\psi(x_1,\cdots,x_N)|^2 dx_1\cdots dx_N. \end{aligned}$$

- System of N bosons interacting via linear potential in the null coordinates.

Hartree Approximation

- What are these bosons?

- They are the **valence-quarks of the parton model**, whose dependence on colour (a_k) has been factored out.

$$\tilde{\Psi}(a_1,p_1;a_2,p_2;\cdots;a_N,p_N) = \epsilon_{a_1,a_2,\cdots a_N}\tilde{\psi}(p_1,p_2,\cdots,p_N).$$

- In the Hartree approximation of atomic and condensed matter physics, the ground state of a many boson system is well approximated by a product

$$\tilde{\psi}(p_1,p_2,\cdots,p_N) = 2\pi\delta(\sum_i p_i - P)\tilde{\psi}(p_1)\tilde{\psi}(p_2)\cdots\tilde{\psi}(p_N).$$

- Hartree approximation to interacting valence parton model is the same as classical Hadron theory on rank 1 sub-manifold of phase space.

Baryon form factor and mass in $N \to \infty$ limit

- Exact baryon form factor in chiral $(m = 0)$ limit is $\tilde{M}(p,q) = -2\psi(p)\psi^*(q)$

$$\tilde{\psi}(p) = \sqrt{\frac{2\pi}{\bar{P}}} e^{-p/2\bar{P}}, \quad p > 0, \quad \bar{P} = P/N.$$

- For small current quark mass, $\tilde{\psi}(p) \sim p^a$ for small p with $a = \sqrt{\frac{3}{\pi}} \frac{m}{\tilde{g}}$

- At $N = \infty$, in the chiral $m = 0$ limit, the lowest lying meson and baryon are massless.

- In the large-N limit for small current quark mass we estimate the square of the baryon mass

$$\mathcal{M}^2 = N^2 \tilde{g}^2 \; [\, \sqrt{\frac{\pi}{3}} \frac{m}{\tilde{g}} + \frac{m^2}{\tilde{g}^2} + \cdots \,]$$

Beyond the Rank 1 Ansatz

- How do we go beyond this valence parton approximation?

- **Restrict** the classical dynamics to larger **submanifolds of phase space**.

- Eg. Rank r ansatz $M_r = \Sigma_{a,b=1}^{r} \xi^a_b \; \psi_a \otimes \psi^{\dagger b}$

- ψ_a orthonormal eigenvectors of ϵ: $\epsilon\psi_a = \epsilon_a \psi_a$, $\epsilon_a = \pm 1$

- M lies on grassmannian if $r \times r$ matrix ξ is hermitian and

$$\xi^b_a \xi^c_b + [\epsilon_a + \epsilon_c]\xi^c_a = 0$$

- ξ lies on a mini Grassmannian!

- **rank 3** ansatz $\rightarrow$ baryon made of **'valence', 'sea' and 'anti-quarks'**.

Baryon form factor models x_B dependence of deep inelastic structure functions

- The quark and anti-quark distributions can be used to model non-perturbative x_{Bj} dependence of nucleon structure functions measured in deep inelastic scattering.

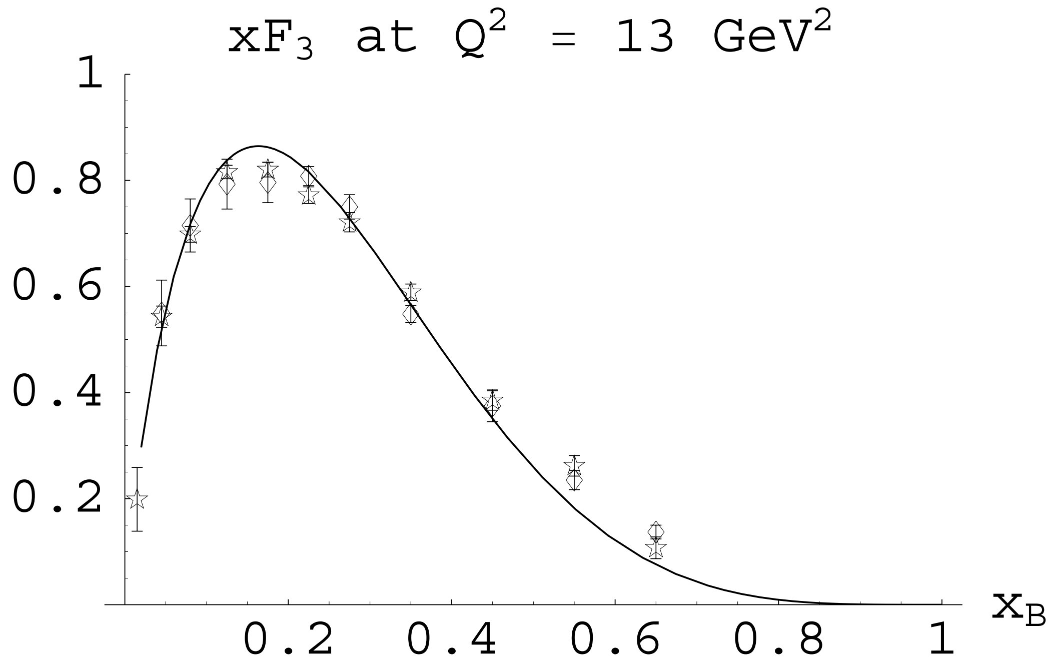

Figure 3: Comparison of predicted xF_3 at $Q^2 = 13$ GeV^2 (solid curve) with measurements by CCFR (star) at 12.6 GeV^2 and CDHS (diamond) at $12.05 \leq Q^2 \leq 14.3$ GeV^2. $Q_0^2 = 0.4$ GeV^2 and $f = \frac{1}{2}$.

- Free parameters: initial value Q_0^2 & fraction of baryon momentum in valence quarks.

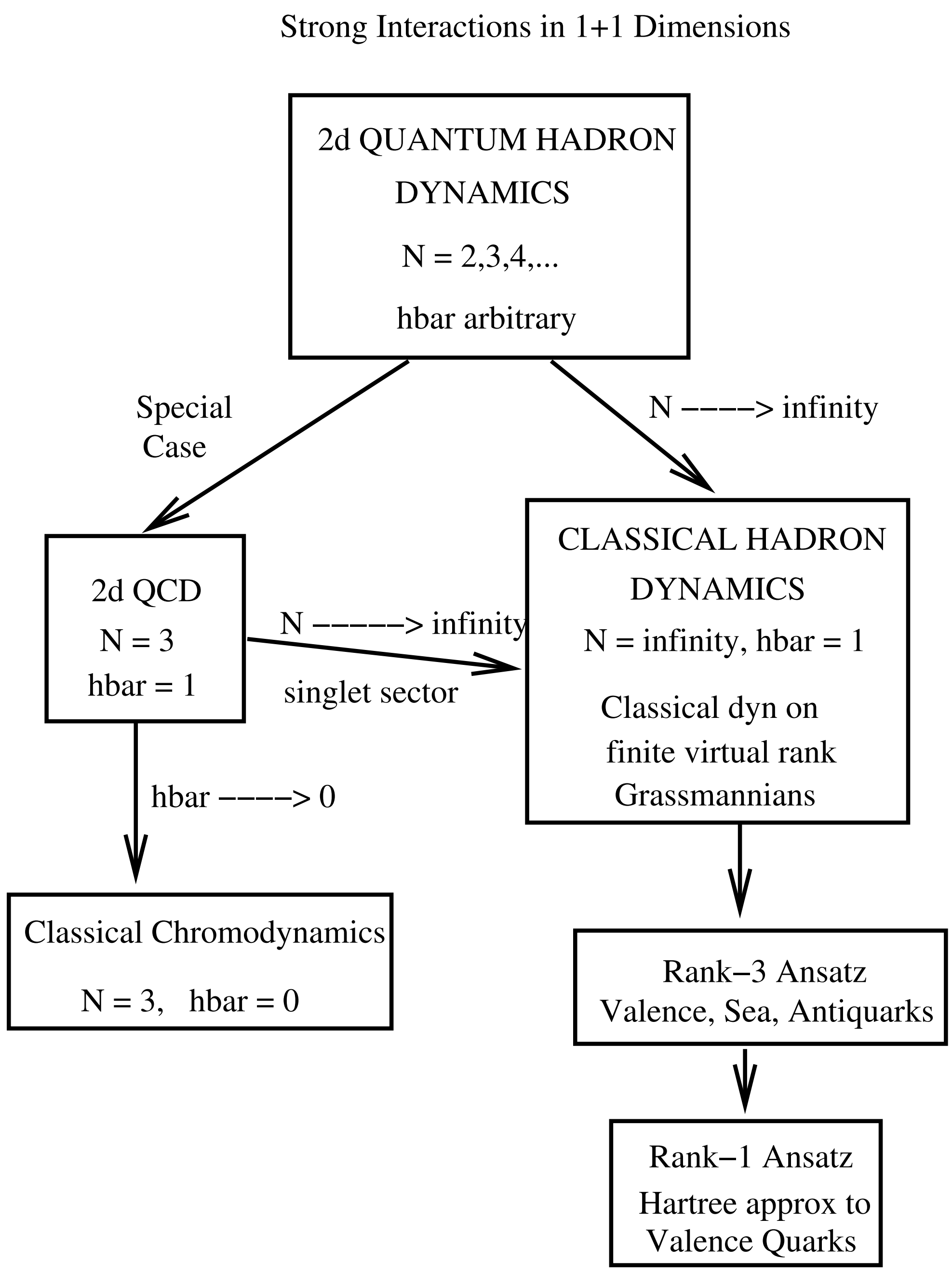
Strong Interactions in 1+1 Dimensions
2d QUANTUM HADRON DYNAMICS
N = 2,3,4,...
hbar arbitrary
Special Case
N ––––> infinity
2d QCD
N = 3
hbar = 1
N –––––> infinity
singlet sector
CLASSICAL HADRON DYNAMICS
N = infinity, hbar = 1
Classical dyn on finite virtual rank Grassmannians
hbar ––––> 0
Classical Chromodynamics
N = 3, hbar = 0
Rank–3 Ansatz
Valence, Sea, Antiquarks
Rank–1 Ansatz
Hartree approx to Valence Quarks

Figure 4:

Facts about 2d Quantum Hadron Dynamics

• It is not a new theory of physics nor a low energy effective theory.

• It is equivalent to the large-N limit of 2d QCD for all values of the 't Hooft coupling constant.

• Upon quantization, it is equivalent to 2d QCD for all numbers of colours and energies.

• Controlled way of extracting quantitative non-perturbative information from QCD_2.

• Baryon number is topologically conserved. Baryon arises as topological soliton. But 2d QHD has an infinite number of mesons, unlike the Skyrme or other low energy effective local soliton models.

Facts about 2d Quantum Hadron Dynamics

• $N = \infty$ QCD_2 is not a free field theory. Only its linearization around the vacuum describes an infinite number of free mesons.

• Would be **difficult** to discover the complete theory including baryons **by summing diagrams**. The non-trivial curved phase space would imply vertices of every order.

• 2d QHD does not seem exactly solvable. However, we obtained the ground state baryon form factor exactly in the large N and chiral limits.

• Neither the quark-parton model nor solitons of low energy effective local Lagrangians are adequate to describe baryons in 2d. But solitons of a bilocal theory are sufficient.

Open String Field Theory

• In the language of string theory, the meson operator $\hat{M}(x, y)$ is the **creation operator** for an **open string** stretching from x to y along a null line.

• The **dynamical degrees of freedom** of the string are concentrated at the **end points**.

• Quantum Hadron Dynamics can be regarded as a light-cone gauge fixed **open string field theory** in two target space-time dimensions.

• The classical system on the Grassmannian provides an approximation method to solve the open string field theory.

Open String Field Theory

• Finding an un-gauge fixed version of this theory is an interesting challenge.

• It is also interesting to determine the **string splitting and scattering amplitudes** from the dynamics on the Grassmannian. These are the leading meson decay and scattering amplitudes.

• It is interesting to know what the corresponding String Field Theory in 3 or 4 target space time dimensions is.