Hadron Physics at Threshold

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INTRODUCTION

$\pi\pi$ scattering

We have witnessed a tremendous progress in low-energy hadron physics over the last decades, both, experimentally and theoretically.

I shall Illustrate this fact with $\pi\pi$ scattering.

Why is $\pi\pi$ scattering interesting?

- Whenever strong interactions are involved at low energy, $\pi\pi$ interactions play a crucial role

  Examples

  - Vacuum polarization contribution to $(g-2)_\mu$

    $\Rightarrow$ Need the pion form factor at low q^2

    $\Rightarrow$ Need P-wave phase shift of $\pi\pi$ scattering
  - Final state interactions among pions in hadronic processes

    Examples: $\eta \to 3\pi, K \to 2\pi, 3\pi$

- Lowest resonance in QCD occurs in $\pi\pi \to \pi\pi$ scattering amplitude [σ-pole on second Riemann sheet]

$$M_\sigma = 441^{+16}_{-6}\,\text{MeV}\,,\ \Gamma_\sigma = 544^{+18}_{-25}\,\text{MeV}\,.$$

Caprini, Colangelo, Leutwyler 2006

- Behaviour of $\pi\pi$ amplitude at threshold is related to QCD-vacuum structure

Fuchs,Knecht, Moussallam,Sazdjian, Stern > 1990

- $\pi\pi$ scattering has become accessible to lattice calculations, see talk by S. Aoki

Some notation

- Consider QCD in the isospin symmetry limit $m_u = m_d$, 6 flavours

  F_π=92.4 MeV $\Rightarrow \Lambda_{\text{QCD}}$

  M_π=139.6 MeV $\Rightarrow m_u$

  M_K=493.7 MeV $\Rightarrow m_s$

  Exact values of heavy quark masses do not matter

- A single function $A(s,t,u)$ determines all channels in $\pi\pi$ scattering

- S-wave scattering lengths are amplitudes at threshold:

$$
\begin{aligned}
T^{\pi^+\pi^-\to\pi^0\pi^0} &= 2N(a_2 - a_0) \\
T^{\pi^+\pi^-\to\pi^+\pi^-} &= N(2a_0 + a_2)\ ;\quad N = 32\pi/6
\end{aligned}
$$

scattering lengths: $a_{0,2\leftarrow\text{isospin}}$

Where is the progress?

ROCHESTER CONFERENCE 1960

Yu. A. Batusov, S.A. Bunyatov, V.M. Sidorov, V.A. Yarba:

Reaction	a_0	a_2	$a_2 - a_0$	References
$\pi^- p \to \pi^+ \pi^- n$			-(0.35 ± 0.30)	Batusov et al., 1960
$\pi N \to \pi N$	~ 1			Efremov et al., 1960
$\pi N \to \pi N$	~ 1			Ishida et al., 1960
$K^{\pm} \to 3\pi$	-0.8	-0.48	0.3	Sawyer and Wali, 1960
$K^{\pm} \to 3\pi$	-1	-0.3	0.7	Khuri and Treiman, 1960
$K^+ \to 3\pi$		~ 1		Thomas et al.,1959

50 YEARS LATER

Theory

$$a_0 = 0.220(5),\ a_0 - a_2 = 0.265(4)$$

$$a_2 = -0.0444(10)$$

Roy+ChPT:Colangelo, J.G., Leutwyler 2000

$$a_2 = -0.04330(42)$$

NPLQCD 2007

Experiment

K_{e4} decays:

$$a_0 = 0.220 \pm 005_{\mathrm{stat}} \pm 0.002_{\mathrm{syst}} \pm 0.006_{\mathrm{theo}}$$

Cusp in $K^+ \to \pi^+\pi^0\pi^0$:

$$a_0 - a_2 = (0.266 - 0.268) \pm 0.003_{\mathrm{stat}} \pm 0.002_{\mathrm{syst}} \pm 0.001_{\mathrm{ext}} \pm 0.013_{\mathrm{theo}}$$

NA48/2 [B. Bloch-Devaux, Confinement 2008; A. Norton, this conference]

Pionium decay: $A_{\pi^+\pi^-} \to \pi^0\pi^0$

DIRAC 2005

$$a_0 - a_2 = 0.265^{+0.033}_{-0.022}$$

Other determinations

	[1]	[2]	[3]
a_0	0.228 ± 0.032	0.224 ± 0.013	0.230 ± 0.015
$-10 \cdot a_2$	0.382 ± 0.038	0.343 ± 0.036	0.480 ± 0.046

S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern 2002 [1]
R. Kaminski, L. Lesniak and B. Loiseau 2003 [2]
J. R. Pelaez and F. J. Yndurain 2005 [3]

See talk Colangelo at KAON07 for comments.

On the use of effective field theories

- A substantial part of the progress on the theory side is due to the use of Effective Quantum Field Theories (EFT).
- It turned out that EFT are also of central importance in data analysis.

  Examples:
  - Radiative corrections in hadron physics.
  - Extrapolation in lattice calculations, both, in volume and in quark mass.

I shall first illustrate the use of EFT by examples (as requested by Hari Dass), and then discuss the construction and the use of Non-relativistic Effective Field Theories (NREFT).

EFFECTIVE FIELD THEORIES

An effective quantum field theory

- contains the degrees of freedom to describe physical phenomena below a chosen energy scale
- ignores the degrees of freedom at higher energies

EFTs are the counterpart of the Theory of Everything.

Ecker 2005

Consider three examples

1. Beta decay

Effective quantum field theories are as old QFT:
β-decay of neutron:

$$n \to p + e^- + \bar{\nu}_e$$

$$\mathcal{L}_{\rm eff} = \text{G}\, \bar{p}\gamma^\mu n \bar{e}\gamma_\mu \nu \text{ + h.c.}$$

Fermi 1933,1934

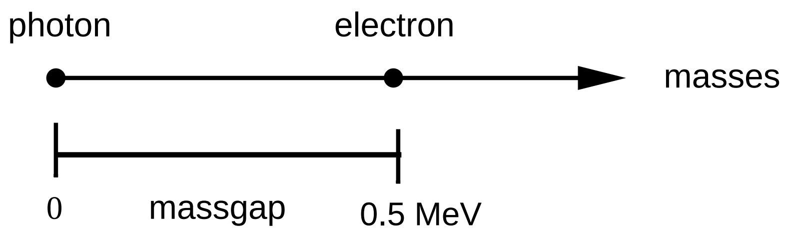
This is an effective, non–renormalizable QFT for the weak interactions at low energies (with due addition of further couplings).

no W, Z

Underlying renormalizable theory: Weinberg 1967; Salam 1968

2. Photon–photon interactions in QED

Consider interaction between photons at very low energies



- Effective Lagrangian for photon interactions: Write all terms allowed by symmetry (gauge, Lorentz, P, C, T)

$$\mathcal{L}_{\rm eff} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{e_1}{m_e^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{e_2}{m_e^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \cdots$$

- Amounts to an expansion in powers of ∂_μ / m_e and $F_{\mu\nu}/m_e^2$
- Scale: electron mass
- Low energy constants (LECs) e_i fixed through QED

Euler, Heisenberg 1936

- $\mathcal{L}_{\rm eff}$ is a non–renormalizable QFT

With properly chosen coefficients e_i, the above Lagrangian reproduces the matrix elements

$$n\gamma \to m\gamma$$

in full QED, to any order in α and in momenta/m_e (for small momenta). This example contains all the bits and pieces relevant for effective Lagrangian techniques:

- Constructing the Lagrangian $\Leftrightarrow$ symmetries
- Loop calculations $\Leftrightarrow$ unitarity
- Power counting $\Leftrightarrow$ calculations become systematic
- Matching $\Leftrightarrow$ relation to underlying theory
- Validity of expansion

3. Effective field theory of QCD

Systematic determination of the structure of matrix elements, as dictated by chiral symmetry:

$$\mathcal{L}_{\mathrm{QCD}} \quad \xrightarrow{E \ll M_\rho} \quad \mathcal{L}_{\mathrm{eff}}$$

$$\boxed{\mathcal{L}_{\mathrm{eff}}}$$

– expressed in observed hadron fields
– same symmetry as QCD

Weinberg 1979

The transition

$$\mathcal{L}_{\mathrm{QCD}} \Longrightarrow \mathcal{L}_{\mathrm{eff}}$$

is non–perturbative $\stackrel{!}{\Longleftrightarrow}$ QED

• The leading term

- Consider pions only:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}_2 + \mathcal{L}_4 + \cdots \\ \mathcal{L}_2 &= \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + M^2(U + U^\dagger)\rangle \\ & \quad U \in SU(2) \ \ , \ \ \text{contains the pion fields}\end{aligned}$$

- $M^2 = (m_u + m_d)B$
- Is derivative expansion
- F, B: low–energy constants (LECs): not fixed by chiral symmetry

$\mathcal{L}_2$: Euler–Heisenberg Lagrangian for mesons – is a non renormalizable quantum field theory

Note: $\mathcal{L}_2$ generates all leading terms for $n\pi \to m\pi$, in contrast to the EH Lagrangian for $n\gamma \to m\gamma$.

• Higher order Lagrangians

$$\mathcal{L}_4 \quad = \quad \sum_{i=1}^{10} \ell_i Q_i \; ; \; \mathcal{L}_6 = \sum_{i=1}^{56} c_i P_i \; ; \; \ldots$$

- LECs ℓ_i, c_i not fixed by symmetry. Local polynomials Q_i, P_i (expressed in meson fields) are known.

  J.G., Leutwyler 1985; Bijnens, Colangelo, Ecker 1999

- Calculations with $\mathcal{L}_{\text{eff}}$ generate an expansion in powers of quark masses and of external momenta. Scale: $4\pi F \simeq$ 1 GeV.

Chiral perturbation theory (ChPT)

$$M_\pi^2 = M^2 \left[1 - \frac{M^2}{32\pi^2 F^2} \bar{\ell}_3 + O(M^4) \right] \; ; \; (MF)^2 = -(m_u + m_d) \langle 0 | \bar{u} u | 0 \rangle$$

$\bar{\ell}_3$ is UV finite part of ℓ_3. Similar keywords as in QED, page 15.

Advantages/disadvantages

Pros

Calculating with this Lagrangian and properly chosen LECs, one reproduces S-matrix elements of QCD, at low energy, in a systematic manner

Weinberg 1979; Leutwyler 1994

Contras

Limited energy range of validity

Many LECs

Note, however: LECs are fixed by QCD. Can be determined either from experiment or using lattice calculations.

$\bar{l}_3$ from lattice

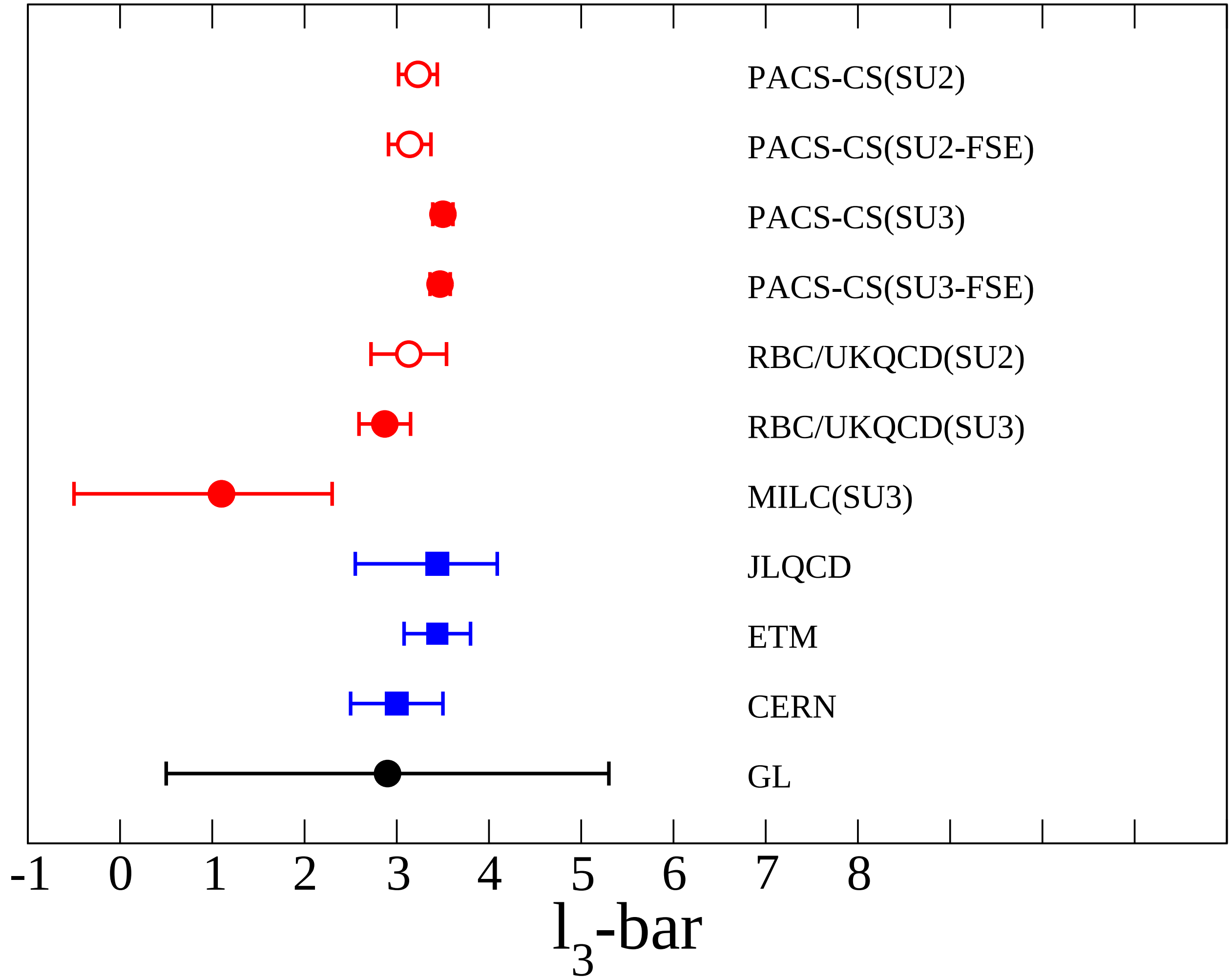


Figure from PACS-CS arXiv: 0807.1661

Applications

MANY

see SPIRES, hep-search

In particular: Including in the effective Lagrangian also nucleon fields, one enters the rich field of nuclear physics.

PRECISION PHYSICS

General comment

- Calculations in theory as well as experiments are performed in our days with very high precision.
- One needs to specify the framework which is used, before a meaningful comparison between theory and experiment can be done.

The paradise world

Predictions for $a_{0,2}$ were made in QCD (6 flavours), with

$$m_u = m_d = m, m_s, \Lambda_{\text{QCD}},$$

chosen such that

$$M_\pi = 139.6\,\text{MeV}\;,\; M_K = 493.7\,\text{MeV}\;,\; F_\pi = 92.4\,\text{MeV}$$

Precise values of heavy quark masses are irrelevant here.
No photons: $F_{\mu\nu} = 0$

Lattice calculations of $a_{0,2}$ can be performed in this framework ($N_f = 2, 3$).
See S. Aoki's talk.

Experiments: phases, cusps, pionium lifetime

In the paradise world:

Pionium does not form (no photons)

The kaon is stable (strangeness conservation): $K \not\to 3\pi, \pi\pi\ell\nu$

$\Rightarrow$ A direct comparison with experiment is not possible – have to enlarge the framework

Experiments

Experiments are performed in the real world, described by the Standard Model

$$\alpha \neq 0\,,\ F_{\mu\nu} \neq 0\,,\ m_u \neq m_d$$

Pionium does form

However:

$K^+ \not\to \pi^+\pi^0\pi^0$
$K^+ \not\to \pi^+\pi^-e^+\nu_e$

ZERO probability that these processes occur in the laboratory: photons are always emitted.

Bloch-Devaux, Nordsieck 1937

$\Rightarrow$ Need a careful analysis of the situation

HOW?

In some cases, proper method is

NREFT

“Non–relativistic effective field theory”

Illustrate it here with pionium decay

Pionium I

Pionium: bound state of $\pi^+\pi^-$. For purely Coulombic bound state:

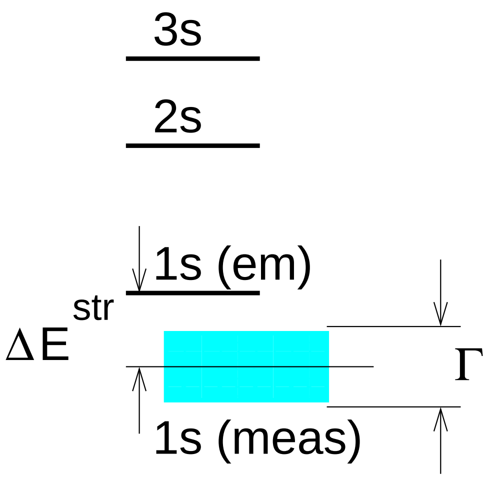
$$E_n = 2M_\pi - \frac{\alpha^2 M_\pi}{4n^2}\,,\; n = 1, 2, \ldots\,.$$

Spectrum is, however, perturbed:

- Higher order corrections in α
- Strong interactions:
  - Energy levels shifted
  - $\pi^+\pi^- \longrightarrow \pi^0\pi^0$

    $\Rightarrow$ Atom becomes unstable, $A_{\pi^+\pi^-} \longrightarrow \pi^0\pi^0$

Scales



- Bohr radius $\simeq 400$ fm
- Lifetime $\tau \simeq 3 \times 10^{-15}$ sec
- $E_2 - E_1 \simeq 1.5 \times 10^3$ eV
- $\Delta E^{\text{str}} \simeq -3.8$ eV , $\Gamma \simeq 0.2$ eV

Effect of strong interactions is very small
with respect to $E_2 - E_1$

DGBT-formula

$$\Gamma_{2\pi^0} = \frac{2}{9}\alpha^3 p^\star (a_0 - a_2)^2 + \cdots \qquad (1)$$

Deser, Goldberger, Baumann, Thirring 1954

Looks very complicated: a_0, a_2 are the scattering lengths in the paradise world. Pionium is bound state in the SM. Isospin broken. How about $+\cdots$ in Eq. (1)?

- Should one apply the Bethe-Salpeter equation?
- Or use potential models?

Clue: $p_{av} \sim r_B^{-1} \sim \alpha\mu_c$: pions are non–relativistic

Only very small momenta count!

$\Rightarrow$ Use non–relativistic QFT

Caswell, Lepage for QED 1986

QCD+QED

$\Downarrow$

NREFT

expansion in

momenta, e

scale M_π

QCD+QED

$\Downarrow$

ChPT

expansion in

momenta,quark masses,e

scale $4\pi F_\pi$

NREFT and ChPT do not talk to each other at the beginning:
different expansions used

Illustration: $\pi\pi \Rightarrow \pi\pi$

Consider partial wave (S-wave, isospin symmetric case)

$$\text{Re } t = a_0 + b_0\,\mathbf{q}^2 + c_0\,\mathbf{q}^4 + \cdots$$

scattering length a_0, effective range b_0, etc

Invoke ChPT

$\Rightarrow$ Expansion of a_0, b_0, c_0 in powers of pion masses

$$a_0 = \frac{7M_\pi^2}{16\pi^2 F^2} + O(M_\pi^4)$$

Weinberg 1966

tree graphs, 1 loop, 2 loops, 3 loops $\cdots$ all contribute

$a_0, b_0, c_0, \ldots$ calculable

Invoke NRQFT

$\Rightarrow$ Can be set up such that

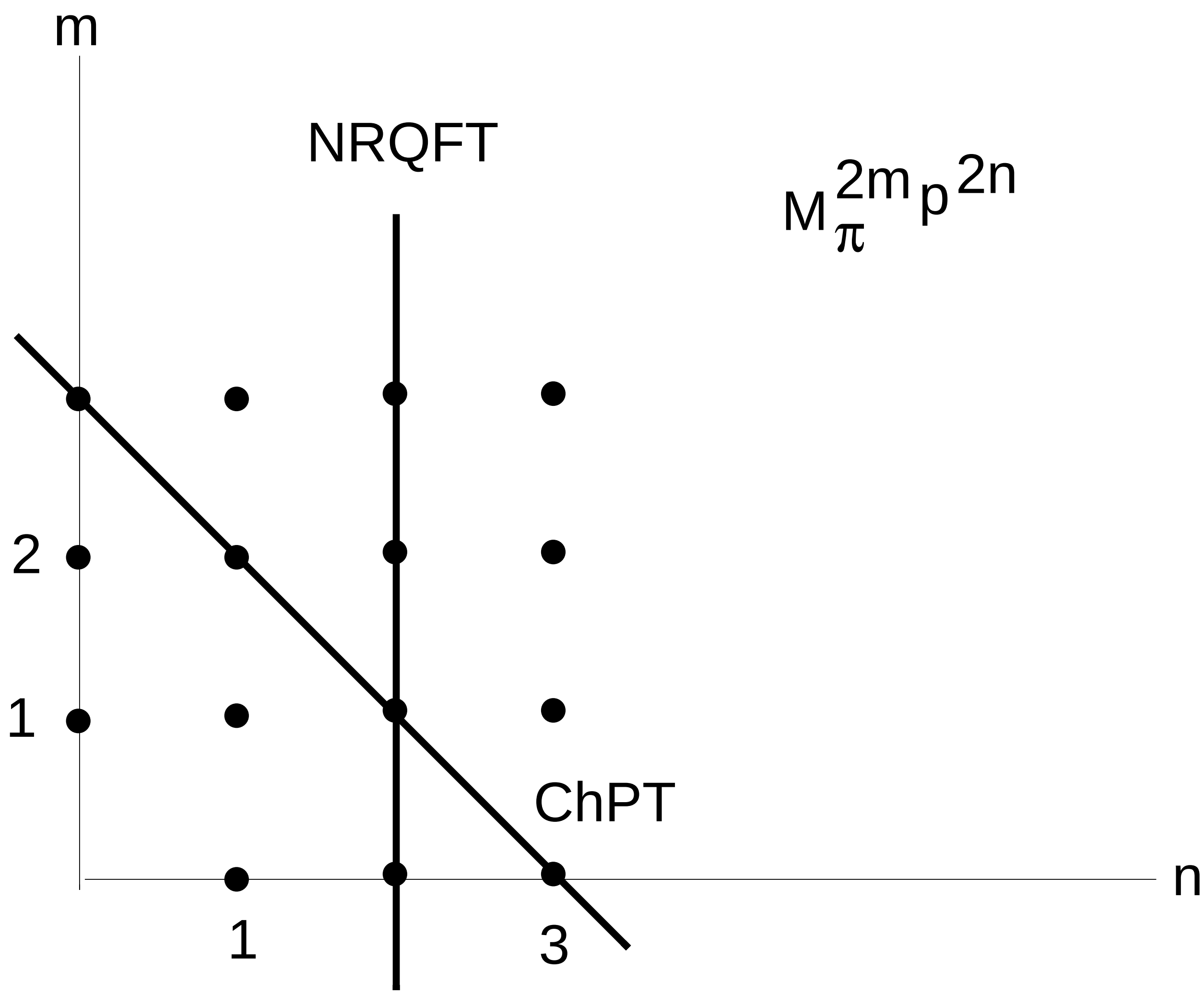
$$\text{Re } t = a_0 + b_0\,\mathbf{q}^2 + c_0\,\mathbf{q}^4 + \cdots$$

with

$$\begin{aligned} a_0 &\Leftarrow \text{tree graph} \\ b_0 &\Leftarrow \text{tree + 2 loops} \\ c_0 &\Leftarrow \text{tree + 2 loops + 4 loops} \end{aligned}$$

a_0, b_0, c_0 are parameters in the theory, cannot be calculated

NRQFT ⇔ ChPT



Why is this useful?

In case that one wishes to extract basic matrix elements from experimental data:
Determine the structure of these with NRQFT

$\Rightarrow$ matrix elements with all the bells and whistles of QFT

Parameters in NRQFT are physical parameters of the underlying theory

Pionium II

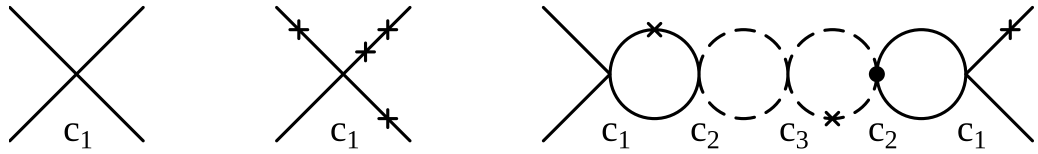
Consider first pions in the absence of electromagnetic interactions.
Symmetries: spatial rotations, P,T. Relevant NR lagrangian is

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_D + \mathcal{L}_S \,, \\
\mathcal{L}_0 &= \sum_{i=\pm,0} \pi_i^\dagger \left(i\partial_t - M_{\pi^i} + \frac{\triangle}{2M_{\pi^i}} \right) \pi_i , \\
\mathcal{L}_D &= \sum_{i=\pm,0} \pi_i^\dagger \left(\frac{\triangle^2}{8M_{\pi^i}^3} + \cdots \right) \pi_i , \\
\mathcal{L}_S &= c_1 \pi_+^\dagger \pi_-^\dagger \pi_+ \pi_- + c_2 [\pi_+^\dagger \pi_-^\dagger (\pi_0)^2 + \mathrm{h.c.}] \\
&\quad + c_3 \, (\pi_0^\dagger \pi_0)^2
\end{aligned}
$$

No other interactions (in 2 particle sector) without spatial derivatives
$c_{1,2,3}$: LECs

Comments

- It is consistent to stay in 2 particle Fock space
  $\pi\pi$ scattering:



- generates expansion in $\sqrt{1-4M^2/s}, M = M_\pi, M_{\pi^0}$
- power counting
- isospin symmetry hidden in the c_i
- illustration: let $M_{\pi^0} = M_\pi$ and go to centre of mass system
  $\Rightarrow$ loop graphs vanish at threshold

Use LSZ to match scattering matrix elements to relativistic theory

Result:

$$4M_\pi^2 c_1 \quad = \quad T_{\mathrm{thr}}^{\pi^+\pi^-\to\pi^+\pi^-}$$: True for any underlying theory

Consider QCD in isospin symmetry limit:

$$3M_\pi^2 c_1 \quad = \quad 4\pi(2a_0 + a_2)$$

This relation is true to *all* orders in the chiral expansion.
Where is chiral symmetry?

Including photons

- Include electromagnetic interactions
- calculate bound state energies in terms of NR LECs
  - Use Coulomb gauge
    $\boldsymbol{H = H_0 + H_C + V}$
  - Count $\alpha, (m_d - m_u)^2$ as terms of order δ

    Expand energy levels in powers of δ
- match NR LECs to relativistic theory
- evaluate isospin breaking corrections in ChPT

Final result

$$\Gamma_{2\pi^0} = \frac{2}{9}\alpha^3 p^* (a_0 - a_2 + \epsilon_1)^2 (1 + K) + o(\delta^{9/2})$$

J.G., Lyubovitskij, Rusetsky, Gall 2001

ϵ_1 is $O(\delta)$, known from ChPT in terms of LECs

$K = K(M_\pi, M_{\pi^0}, a_0, a_2, \alpha)$ is known, order O(δ)

p^* momentum of outgoing pions

Scattering lengths from ChPT + Roy $\Rightarrow$

$$\tau = (2.9 \pm 0.1) \cdot 10^{-15} \text{ sec}$$

$$\tau = (2.91^{+0.49}_{-0.62}) \cdot 10^{-15} \text{ sec}$$

DIRAC 2005; analysis ongoing

Other hadronic atoms

$\pi^+\pi^-$ atom	:	pionium
πK atom		
$\pi^- p$ atom	:	pionic hydrogen
$\pi^- d$ atom	:	pionic deuterium
$K^- p$ atom	:	kaonic hydrogen
$K^- d$ atom	:	kaonic deuterium

The same technique can be used for all of these.

J.G., Ivanov, Lipartia, Lyubovitskij, Meißner, Mojžiš, Raha, Rusetsky,
Schärli, Schweizer, Zemp,. . .
J.G., Lyubovitskij, Rusetsky, Phys. Rep. 2008

NREFT:

$K \to 3\pi$ decays

See also talk by A. Norton, this conference

Threshold Anomalies

P. Budini, L. Fonda:

$\pi\pi$ interactions from threshold anomalies in K^+decays

Phys. Rev. Lett. 1961

one year after Rochester 1960 conference

They say:

The method is based on the observation of threshold anomalies of the kind of a cusp or rounded step in cross sections for reactions with three particles in the final state.

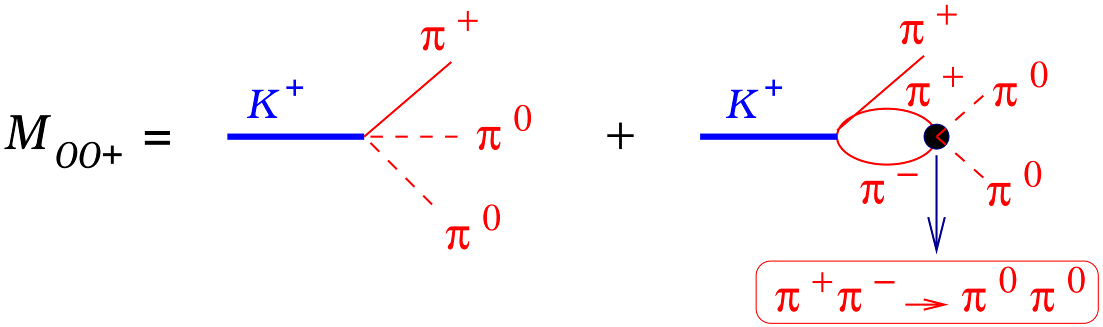
As an example, we will discuss the decay of the K^+ meson into 3 pions, for which all the requirements for the appearance of the effect are satisfied:

$$K^+ \to \pi^+\pi^-\pi^+$$

$$K^+ \to \pi^+\pi^0\pi^0$$

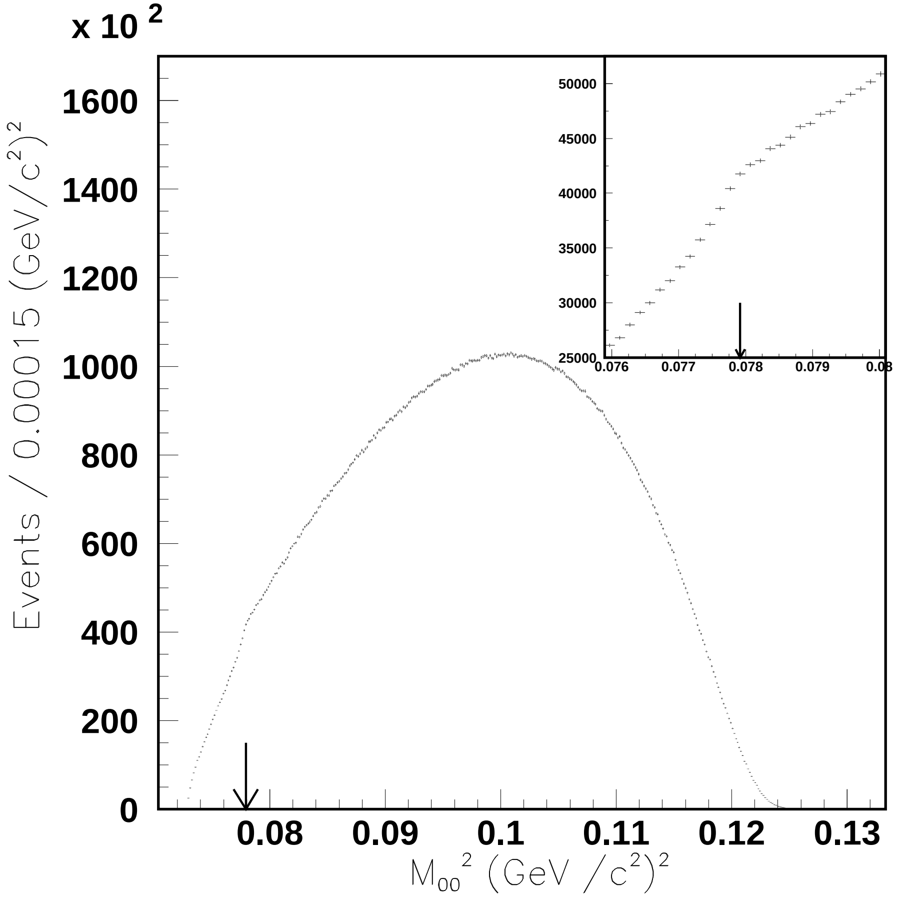
After a very beautiful explanation:

We see, therefore, that if a detailed measurement of the π^+ distribution is performed around the $\pi^+\pi^-$ threshold, one can determine in principle, with the use of this equation, the pion-pion charge exchange cross section at zero energy.



14 years after discovery of π [Lattes et al., 1947] (!)

Today



$K^+ \to \pi^+\pi^0\pi^0$

Partial sample of $\sim 2.3 \cdot 10^7$ decays

J. R. Batley *et al.* [NA48/2 Collaboration], PLB 633 (2006) 173

Comments

- Investigations based on original idea of Wigner 1948: “Wigner cusp”
- A large body of investigations of the Wigner cusp around 1960:
  Breit 1957; Newton 1958; Adair 1958; Fonda,Newton 1959; Fonda 1959; Baz 1958, 1959; Budini, Fonda 1961; . . .
  The authors investigated cross sections, decay processes.
- These facts were largely forgotten, then rediscovered by Meißner, Müller, Steininger 1997 [$\pi\pi \to \pi\pi$] and by Cabibbo 2004 [$K \to 3\pi$]
- Budini and Fonda had no data available back in 1961.
  NA48/2: tens of million events — Norton, this conference
- Data are now so precise that radiative corrections need to be taken into account, both in $K \to 3\pi$ and in K_{e4} decays.

In the following , we construct NREFT for these decays $\Rightarrow a_{0,2}$

Power counting

Consider a world with $m_s < 20\,\hat{m}$
pion momenta in $K \to 3\pi$ very small
Count kinetic energies as small

$$T_i = p_i^0 - M_{\pi_i} = \mathcal{O}(\epsilon^2)$$

Then

$$M_K - \sum_i M_{\pi_i} = \mathcal{O}(\epsilon^2)$$

for consistency reasons.

- Additional small parameter: $\pi\pi$ scattering lengths.
- Expand amplitude in powers of ϵ, a.

Cabibbo 2004; Cabibbo, Isidori 2005

- To a given order in a, ϵ, a well defined finite number of graphs contributes.
- This feature is called a "power counting scheme".
- Additional good feature: Photons can be included in a straightforward manner - this is not the case in an S-matrix framework.

The lagrangian

$$\mathcal{L}_{NR} = \mathcal{L}^{kin} + \sum_i \mathcal{L}_{\pi\pi}^{(i)} + \sum_i \mathcal{L}_{K3\pi}^{(i)}$$

Non relativistic framework: Pion propagators is

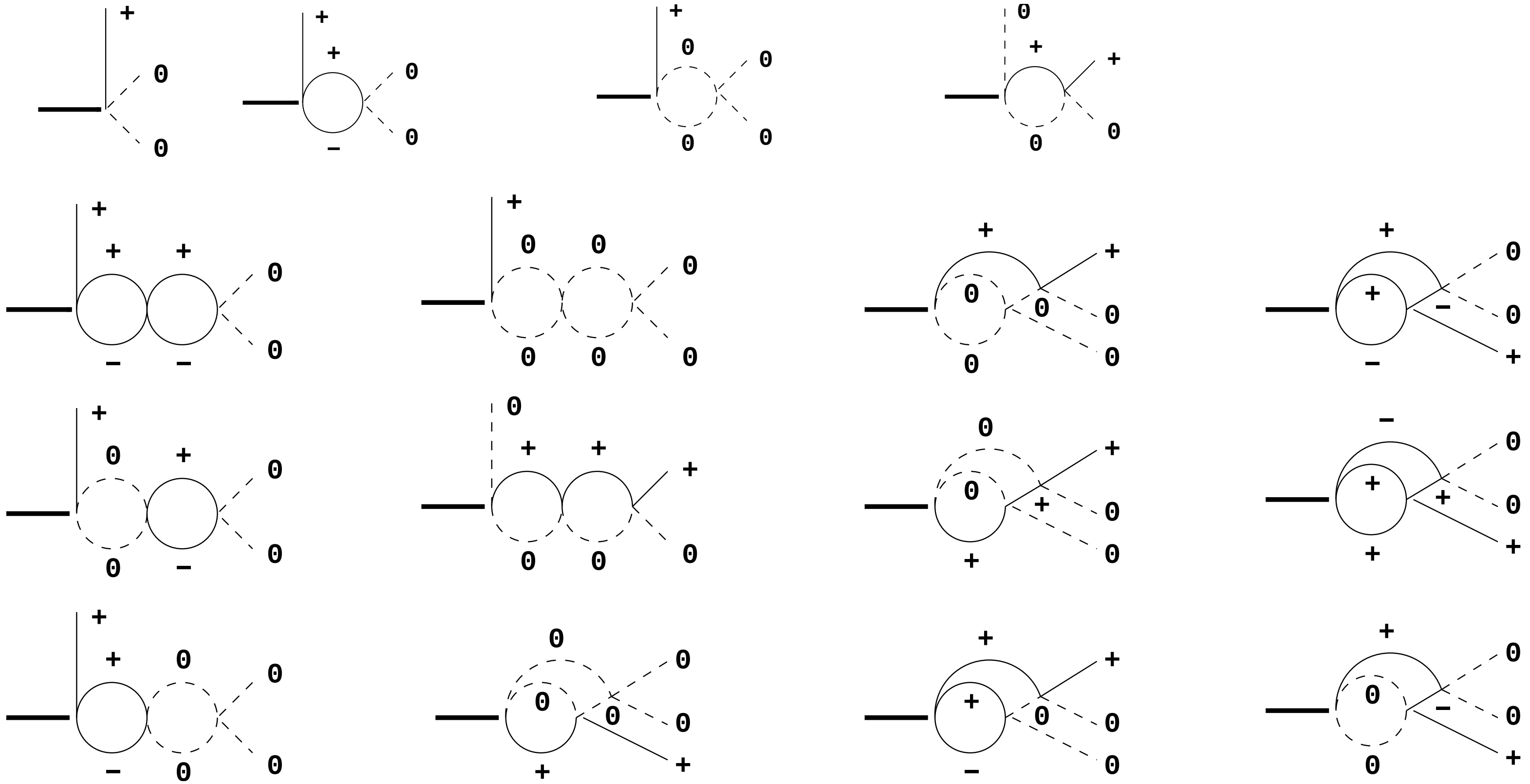
$$\Delta_\pi(p^0, \mathbf{p}) \quad = \quad \frac{1}{\omega(\mathbf{p})(\omega(\mathbf{p}) - p^0)}\,, \qquad \omega(\mathbf{p}) = (M_\pi^2 + \mathbf{p}^2)^{1/2}$$

Interactions:

$$\mathcal{L}_{\pi\pi}^{(1)} \quad = \quad c_0\big(\pi_-^\dagger \pi_+^\dagger \pi_0 \pi_0 + h.c.\big)$$

$$\vdots$$

The graphs



Evaluate these with NR propagators.

Result

$$\mathcal{M} = \underbrace{P_0 + P^T J}_{Cabibbo 2004} + J^T C J + D^T F + \mathcal{O}(\epsilon^6, a\epsilon^7, a^2\epsilon^6, a^3\epsilon^3)$$

where

$J = (J_1, ..., J_N)$: one loop integrals $O(\epsilon)$

$F = (F_1, ..., F_M)$: two loop integrals $O(\epsilon^2)$

[in explicit form]

and

$C : N \times N$ matrix; $D = (D_1, .., D_M)$

$P_0;\ P = (P_1, ..., P_N)$

} polynomials

Bissegger, Colangelo, Fuhrer, J.G., Kubis, Rusetsky > 2006

- Above representation valid in decay region and slightly beyond.
- Contains of course all cusps in the decay region and on its boundaries.
- Satisfies all holomorphic properties.
- Radiative corrections have recently been included.

  Bissegger, Fuhrer, J.G., Kubis, Rusetsky 2008
- Representation presently used by NA48/2 collaboration for data analysis

OTHER APPROACHES

1. Gribov; Anisovich, Ansel'm
   - Series of articles on threshold singularities in late 50's, early 60's. Expansion up to a^3.
   - Present framework: QFT realization of their approach.
   - Anisovich, Ansel'm: provide a proof that there is no anomalous threshold.

2. Cabibbo 2004; Cabibbo and Isidori 2005
   - Our result agrees algebraically with CI at one loop order.
   - At two loops, our result agrees at the threshold for $\pi\pi$ production. Away from this threshold, the result differs.

Reason for the difference: At one loop, the decay amplitude can be written as

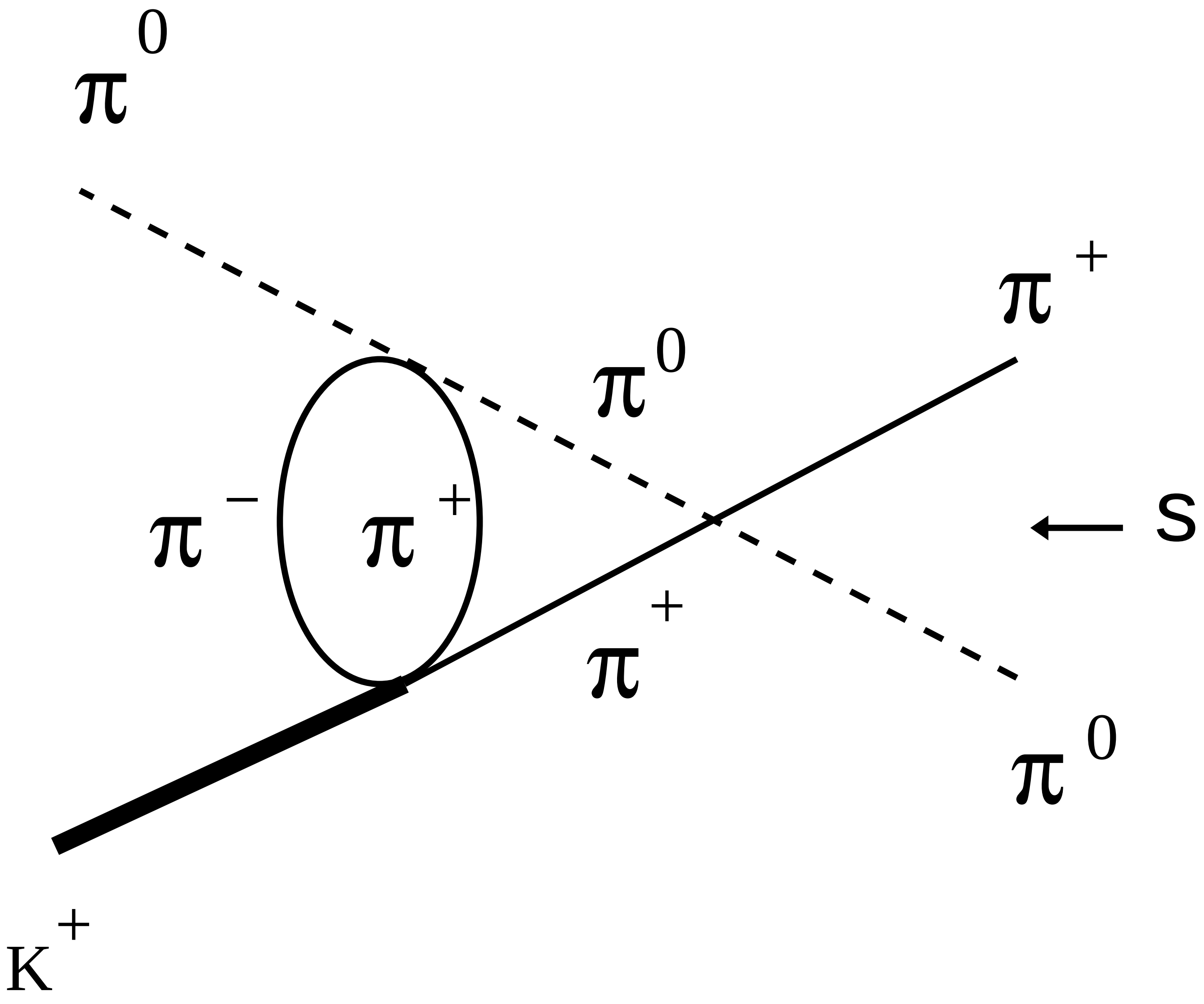
$$\mathcal{M} = A + vB \qquad [\text{order } a] \qquad (^{***})$$

Second term generates cusp. A, B are analytic functions in decay region.

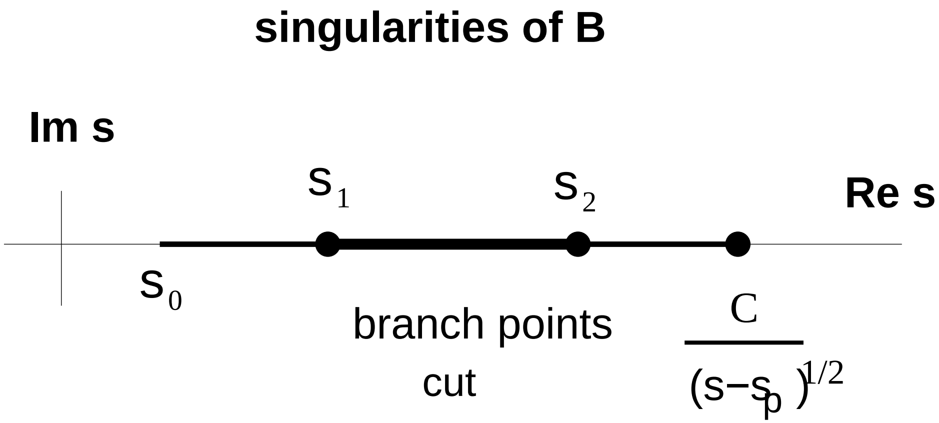
- CI assume that $(^{***})$ is a valid decomposition in general. Use then approximation to calculate A, B.
- In contrast, we perform a complete two-loop calculation of $\mathcal{M}$.

Beyond one loop, decomposition $(^{***})$ is not possible in whole decay region with analytic A, B.

Example



$$\mathcal{M} \quad = \quad A + \bar{v}B$$



$s_0^{1/2} = 274$ MeV $s_1^{1/2} = 308$ MeV

Singularities at $s_{1,2,p}$ absent in $\mathcal{M}$ on upper rim of the cut.

3. Gamiz, Prades, Scimemi 2006.
Combination of ChPT and dispersion relations. Is not a systematic procedure in my opinion.

4. Kampf, Kecht, Novotny, Zdrahal 2008
Dispersive representation, using ChPT. Performed for $K_L \to 3\pi^0$. I do not see how photons can consistently be included in this framework: [e.g., in the presence of photons, holomorphic properties of amplitude is fundamentally changed (pionium poles).]

STATUS

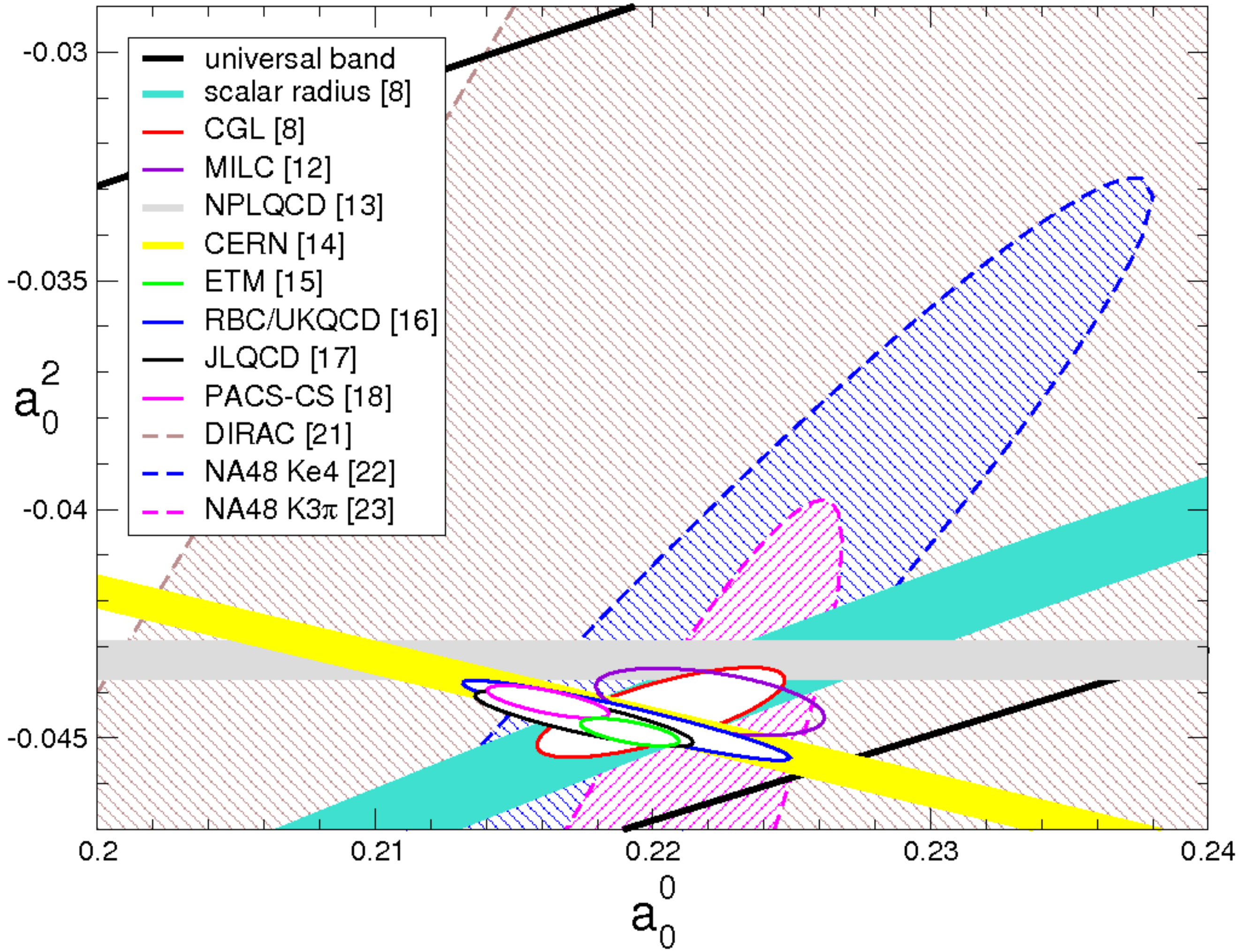


Figure from H. Leutwyler, arXiv:0807.1661

SUMMARY

- Tremendous progress in last decade in precision of low-energy hadron physics, both, in theory and in experiment.
- Lattice calculations are developing very fast, and have come into contact with ChPT.
- At this precision, it is important to include isospin breaking effects in a proper manner.
- Non-relativistic quantum field theories are a very convenient tool in this connection.
- It would be useful to apply this technique to K_{e4} decays, which were not discussed here.