Effective String Theory: Covariant Calculus

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Covariant Calculus for Effective String Theories,
P. Matlock, N. D. Hari Dass, `arXiv 0709.1765`

Things which are in this talk

- String Theory - basic formulation, action, symmetries
- Effective - as usual, but in string context
- Covariant - initial symmetries, reduced to conformal gauge
- Calculus - formulate actions easily

What is not in this talk

- prescription for or direct application to QCD strings

String Theory

- Action is in terms of an embedding into target space

$$S = \int \mathrm{d}^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

I. worldsheet diffeomorphism symmetry: $\sigma \to \sigma'(\sigma)$

II. Weyl-scaling symmetry: $h_{\alpha\beta}(\sigma) \to \omega(\sigma) h_{\alpha\beta}(\sigma)$

Use I to bring the metric h to the form $h = \begin{bmatrix} 0 & \varphi \\ \varphi & 0 \end{bmatrix}$.

Use II to further reduce h to the form $h = 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- Residual symmetry: combinations of I and II which preserve this.

Now, we have coordinates τ^+, τ^- on a flat plane; $\tau^\pm := \tau \pm \sigma$.
The residual symmetry is that of conformal transformations;
action is in *conformal gauge*.

Conformal transformations

- Conformal-gauge action

$$S = \int \mathrm{d}\tau^{\pm} \partial_{+} X \cdot \partial_{-} X$$

is exactly invariant under infinitesimal transformation $\delta_{\pm}$

$$\delta_{+} X(\tau^{+}, \tau^{-}) = \epsilon^{+}(\tau^{+}) \partial_{+} X(\tau^{+}, \tau^{-})$$

and

$$\delta_{-} X(\tau^{+}, \tau^{-}) = \epsilon^{-}(\tau^{-}) \partial_{-} X(\tau^{+}, \tau^{-}).$$

We abbreviate

$$\delta_{\pm} X = \epsilon^{\pm}(\tau^{\pm}) \partial_{\pm} X.$$

Polchinski & Strominger's theory [PRL 67, 1681 (1991)]

- "Begin in and work in conformal gauge.
- "Write down all terms of correct dimension, and drop all EOM ($\sim \partial_\pm X$) terms. (field definition)
- "Expand around the classical background $X^\mu = e^\mu_- R\tau^- + e^\mu_+ R\tau^+$.
- "For a long-string expansion around our particular background, only valid denominator is $L := \partial_+ X \cdot \partial_- X \sim \mathcal{O}(R^2)$.

$$S = \frac{1}{4\pi}\int \mathrm{d}\tau^\pm \left[\frac{1}{a^2}L + \beta\frac{\partial_+^2 X \cdot \partial_- X \partial_+ X \cdot \partial_-^2 X}{L^2} + \cdots \mathcal{O}(R^{-4})\right]$$

- Conformal symmetry is approximate. $\delta_\pm$

$$\delta_- X = \epsilon^-(\tau^-)\partial_- X - \frac{1}{2}\beta a^2 \partial_-^2 \epsilon^-(\tau^-)\frac{\partial_+ X}{L}$$

- In principle this can be extended to higher orders, but: Next-order action $\rightarrow$ New transformation law.

Higher-order P&S in practice

$$S = \frac{1}{4\pi} \int \mathrm{d}\tau^{\pm} \left[\frac{1}{a^2} L + \beta \frac{\partial_+^2 X \cdot \partial_- X \partial_+ X \cdot \partial_-^2 X}{L^2} + \sum_j \gamma_j M_j + \cdots \mathcal{O}(R^{-6}) \right]$$

$$M_1 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_+^2 X \ \partial_-^2 X \cdot \partial_-^2 X,$$

$$M_2 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_-^2 X \ \partial_+^2 X \cdot \partial_-^2 X,$$

$$M_3 = \frac{1}{L^4} \partial_-^2 X \cdot \partial_+^2 X \partial_- X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+ X,$$

$$M_4 = \frac{1}{L^5} (\partial_- X \cdot \partial_+^2 X)^2 (\partial_-^2 X \cdot \partial_+ X)^2$$

[Drummond, `hep-th/0411017`]

- Four new parameters γ_j will be required.
- New transformation law must be found. Does it exist?

Exact conformal symmetry

- Instead of

$$S = \frac{1}{4\pi}\int \mathrm{d}\tau^{\pm}\left[\frac{1}{a^2}L + \beta\frac{\partial_+^2 X\cdot\partial_- X\partial_+ X\cdot\partial_-^2 X}{L^2}\right]$$

add some EOM terms to write

$$S = \frac{1}{4\pi}\int \mathrm{d}\tau^{\pm}\left[\frac{1}{a^2}L + \beta\frac{\partial_+ L\partial_- L}{L^2}\right]$$

- This is now ‘complete’, in the sense of having exact conformal symmetry, $\delta_- X(\tau^+,\tau^-) = \epsilon^-(\tau^-)\partial_- X(\tau^+,\tau^-)$.

- This theory can itself be analysed to any order in $1/R$.

- The goal is to construct a formalism to write all higher-order terms in a way that such symmetry is present and manifest.

Calculation of spectrum

- Action: $S = \frac{1}{4\pi}\int \mathrm{d}\tau^{\pm}\left[\frac{1}{a^2}L + \beta\frac{\partial_+ L \partial_- L}{L^2}\right]$
- EM tensor:

$T_{--} = 2\frac{\partial_- X\cdot\partial_- X}{a^2} + 2\beta\left[(\frac{\partial_- L}{L})^2 - 2\frac{\partial_- X\cdot\partial_- X}{L}\partial_\pm \log L - 2\partial_-^2 \log L\right]$

- Equation of Motion:

$\frac{2}{a^2}\partial_\pm X = 2\beta\left[\partial_+\frac{N\partial_- X}{L} + \partial_-\frac{N\partial_+ X}{L}\right]$ with $N := \partial_\pm \log L$

- Background solution & fluctuation field:

$X = X_0 + Y$ with $X_0^\mu = e_-^\mu R\tau^- + e_+^\mu R\tau^+$.

- Solve Y EOM and write holomorphic form

$T_{--}(\tau^-) = \frac{-R}{a^2}e_-\cdot\partial_- Y - \frac{1}{2a^2} : \partial_- Y\cdot\partial_- Y : -\frac{\beta}{R}e_+\cdot\partial_-^3 Y + \mathcal{O}(R^{-2})$

- Central charge: $c = D + 12\beta - 26$
- Expand in Virasoro modes $T_{--} = \sum_m L_m e^{im\tau^-}$

$L_m = \frac{R}{a}e_-\cdot\alpha_m + \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m : + \frac{a\beta}{R}n^2 e_+\cdot\alpha_n + \mathcal{O}(R^{-2})$

the spectrum can then be obtained in the usual way from L_0.

Covariant treatment of general actions

- Take a hint from usual string theory; origin of conformal invariance is Reparametrisation + Weyl invariance

- Familiar covariant derivative:

$$\nabla_\alpha T_\beta := \partial_\alpha T_\beta - \Gamma^\gamma_{\alpha\beta} T_\gamma$$

Using this derivative, we may write actions which are [manifestly] diffeomorphism invariant.

$$\nabla(\text{tensor}) = \text{a tensor}$$

- We wish to make a similar construction for Weyl scaling.

Weyl-covariant derivative [scalars]

- Under a Weyl-scaling $\omega(\tau^{\pm})$, a field $\phi^{(j)}$ with Weyl-dimension $j = W[\phi]$ transforms as $\phi \to \phi' = \omega^j \phi$.
In particular, $W[h_{\alpha\beta}] = -W[h^{\alpha\beta}] = +1$ and $W[X] = 0$.

- The Weyl-covariant derivative on $\phi^{(j)}$ is given by

$$\Delta_\alpha \phi^{(j)} := \partial_\alpha \phi^{(j)} - j\chi_\alpha \phi$$

where the Weyl connection is

$$\chi_\alpha := W[\Phi]^{-1} \partial_\alpha \log \Phi$$

and Φ is *any scalar field*.

- The name "Weyl-covariant derivative" is appropriate, for

$$W[\Delta_\alpha \phi] = W[\phi].$$

Weyl-covariant derivative [tensors]

- Define a connection

$$G^{\gamma}_{\alpha\beta} := \Gamma^{\gamma}_{\alpha\beta} + W^{\gamma}_{\alpha\beta}$$

which includes the usual Christoffel connection

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta}(\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta})$$

and also the Tensor Weyl-connection

$$W^{\gamma}_{\alpha\beta} := \frac{1}{2}(\chi^{\gamma} h_{\alpha\beta} - \chi_\beta \delta^{\gamma}_{\alpha} - \chi_\alpha \delta^{\gamma}_{\beta})$$

- Now, the Fully-covariant derivative of a tensor may be written

$$\mathcal{D}_\alpha T^{(j)}_{\beta} = \partial_\alpha T^{(j)}_{\beta} - j\chi_\alpha T^{(j)}_{\beta} - G^{\gamma}_{\alpha\beta} T^{(j)}_{\gamma}$$

or more explicitly,

$$\mathcal{D}_\alpha T^{(j)}_{\beta_1\ldots\beta_n} = \partial_\alpha T^{(j)}_{\beta_1\ldots\beta_n} - j\chi_\alpha T^{(j)}_{\beta_1\ldots\beta_n} - \sum_i G^{\gamma}_{\alpha\beta_i} T^{(j)}_{\beta_1\ldots\gamma\ldots\beta_n}$$

Weyl-Covariant Derivative Properties

$$\mathcal{D}_\alpha T^{(j)}_\beta := \partial_\alpha T^{(j)}_\beta - j\chi_\alpha T^{(j)}_\beta - G^\gamma_{\alpha\beta} T^{(j)}_\gamma$$

- Covariance

  $\mathcal{D}$(tensor of Weyl dimension j) = a tensor of Weyl dimension j

- Metric is covariantly constant

$$\mathcal{D}_\alpha h_{\beta\gamma} = 0$$

- Leibniz rule

$$\mathcal{D}_\alpha (T_1 T_2) = (\mathcal{D}_\alpha T_1) T_2 + T_1 (\mathcal{D}_\alpha T_2)$$

$$\mathcal{D}_\alpha \Phi = 0$$

Manifestly invariant actions

- An invariant action:

$$S = \int \mathrm{d}^2\sigma \sqrt{h}\mathcal{M}$$

where $\mathcal{M}$ is a worldsheet scalar with $W[\mathcal{M}] = -1.$

- Manifest diffeomorphism invariance.

- Manifest Weyl-scaling invariance.

- Such an $\mathcal{M}$ may be built covariantly: $\{ \mathcal{D}_\alpha, X^\mu, h_{\alpha\beta}, \Phi \}$

Higher-order invariant actions

$$S = \frac{1}{4\pi}\int \mathrm{d}\tau^{\pm}\left[\frac{\sqrt{h}}{a^2}\mathcal{L} + \beta\frac{\partial_+ L \partial_- L}{L^2} + \sqrt{h}\mu_1\mathcal{M}_1 + \sqrt{h}\mu_2\mathcal{M}_2 + \cdots \mathcal{O}(R^{-6})\right]$$

where the following are Scalars with Weyl dimension -1.

$$\mathcal{L} = h^{\alpha\beta}\mathcal{D}_\alpha X \cdot \mathcal{D}_\beta X$$
$$\mathcal{M}_1 = \frac{1}{\mathcal{L}^3}h^{\alpha\alpha'}h^{\beta\beta'}h^{\gamma\gamma'}h^{\delta\delta'}\mathcal{D}_\alpha\mathcal{D}_\beta X \cdot \mathcal{D}_\gamma\mathcal{D}_\delta X \mathcal{D}'_\alpha\mathcal{D}'_\beta X \cdot \mathcal{D}'_\gamma\mathcal{D}'_\delta X$$
$$\mathcal{M}_2 = \frac{1}{\mathcal{L}^3}(h^{\alpha\alpha'}h^{\beta\beta'}\mathcal{D}_\alpha\mathcal{D}_\beta X \cdot \mathcal{D}'_\alpha\mathcal{D}'_\beta X)^2$$

Conformal gauge-fixing of the action gives exactly the terms we had before. We must choose $\Phi = \mathcal{L}$.

- Only two parameters for the $\mathcal{O}(R^{-4})$ part.
- Every* term is [manifestly] covariant.

Main points

- [Exactly] invariant actions, to any order in the string length R.
- Standard and unchanging conformal transformation law.
- $\rightarrow$ Machinery of CFT may be used in analysis.
- Reduction of the number of parameters in the effective action.

In progress/Future

- Higher-order analysis of invariant version of PS action.
- Analysis of spectrum from $\mathcal{M}$ terms.
- Modification of the theory for slightly more realistic QCD situation.
- $\rightarrow$ Use of different string backgrounds, insertion of quarks, etc.