σ-resonance and convergence of chiral perturbation theory

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References:
Phys. Rev. D77, 014506 (2008)
Phys.Rev. D77, 091501(R) (2008)

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Outline

- Motivation
- Why σ-resonance?
- Features of our model
- Results

Motivation

- Region in quark mass where chiral perturbation theory is valid, is an important question for Lattice QCD but difficult to answer.
- The role of resonances is particularly important.
- The problem is non-perturbative.
- No lattice studies within models exist, where it has been shown how the existence of a resonance limits the region of applicability of ChPT.
- We want to construct a lattice model of pions of two flavor QCD, study the convergence of the chiral expansion in the presence of a light and narrow σ-resonance.

Why σ resonance?

- Since our model is NOT QCD it is difficult to know what resonances exist in it.
- In most models it is possible to tune parameters to create σ as a light and narrow sigma.
- Other resonances are more complex and difficult to create. Perhaps they too can be created and studied?
- Here we focus on the effects of the σ resonance on chiral perturbation theory from first principles.
- Our study will be somewhat indirect as we will see. A more direct study is postponed to the future.

How to create a light and narrow σ-resonance in a model?

- If chiral symmetry is broken in a 4+1 dimensional lattice field theory then
  - heating the theory should restore the symmetry at T_c
  - close to T_c a linear sigma model is a good description
  - in lattice units the pion decay const. goes to zero at T_c

$$F \sim (T_c - T)^{\frac{1}{2}} \quad \text{up to log corrections}$$

  - due to chiral symmetry restoration and logarithmic triviality of the theory at T_c we must have

$$\frac{M_\sigma}{F} \to 0 \quad \frac{\Gamma_\sigma}{F} \to 0$$

By remaining close to T_c on the broken side one must be able to create a light and narrow sigma.

Features of our model

- Model is constructed with fermionic degrees of freedom
- Pions arise from confinement and chiral symmetry breaking just like in QCD.
  - more closer to QCD in spirit
  - has a pedagogical value
- A well defined lattice field theory with all the chiral symmetries of QCD intact.
- Can be studied with efficient "cluster" algorithms in the chiral limit and close to it.

Our Model

d=4

Action

$$S = -\sum_x \sum_{\mu=1}^{d+1} \eta_{\mu,x} \left[e^{i\phi_{\mu,x}} \overline{\psi}_x \psi_{x+\hat{\mu}} - e^{-i\phi_{\mu,x}} \overline{\psi}_{x+\hat{\mu}} \psi_x \right] - \sum_x \left[m\overline{\psi}_x \psi_x + \frac{\tilde{c}}{2} \left(\overline{\psi}_x \psi_x \right)^2 \right]$$

U(1) gauge theory

no gauge action → strongly coupled!

mass

Anomaly

$$\psi_x = \begin{pmatrix} u_x \\ d_x \end{pmatrix}$$ ← Two flavors → $$\overline{\psi}_x = \begin{pmatrix} u_x & d_x \end{pmatrix}$$

$$\left(\eta_{\mu,x}\right)^2 = 1, \mu = 1,2,3,4$$

$$\left(\eta_{5,x}\right)^2 = T$$

fictitious temperature

Lattice size L x L x L x L x L_5

$L_5 = 2$

Model has symmetries of N_f=2 QCD

Symmetries

Our model contains all the chiral symmetries

$$S = -\sum_x \sum_{\mu=1}^{d+1} \eta_{\mu,x} \left[\mathrm{e}^{i\phi_{\mu,x}} \overline{\psi}_x \psi_{x+\hat{\mu}} - \mathrm{e}^{-i\phi_{\mu,x}} \overline{\psi}_{x+\hat{\mu}} \psi_x \right] - \sum_x \left[m\overline{\psi}_x \psi_x + \frac{\tilde{c}}{2} \left(\overline{\psi}_x \psi_x \right)^2 \right]$$

When $m = 0$ and $\tilde{c} = 0$

the action is invariant under $SU(2) \times SU(2) \times U(1)$.

If x is even

$$\psi_x \longrightarrow \mathrm{e}^{i\theta}\ L\ \psi_x,$$

$$\overline{\psi}_x \longrightarrow \overline{\psi}_x\ R^\dagger\ \mathrm{e}^{i\theta},$$

$L\ ,R \in SU(2)$

If x is odd

$$\psi_x \longrightarrow \mathrm{e}^{-i\theta}\ R\ \psi_x,$$

$$\overline{\psi}_x \longrightarrow \overline{\psi}_x\ L^\dagger\ \mathrm{e}^{-i\theta},$$

When $\tilde{c} \neq 0$ then the $U(1)$ symmetry is broken to Z_2.

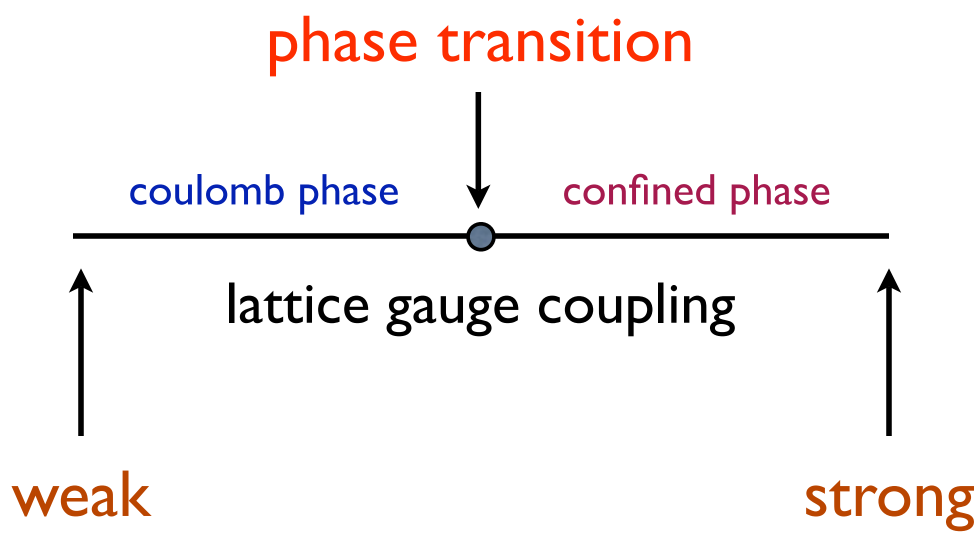
$$\frac{1}{2}(\overline{\psi}_x \psi_x)^2 \equiv \overline{u}_x u_x \overline{d}_x d_x$$

$$\overline{u}_x u_x \overline{d}_x d_x \rightarrow \mathrm{Det}(R^\dagger L \mathrm{e}^{i2\theta}) \overline{u}_x u_x \overline{d}_x d_x$$

$$\overline{u}_x u_x \overline{d}_x d_x \rightarrow \mathrm{Det}(L^\dagger R \mathrm{e}^{i2\theta}) \overline{u}_x u_x \overline{d}_x d_x$$

How can QED show confinement and chiral symmetry breaking?

On the lattice there is a confined phase at strong couplings!



Partition function

The partition function of our model is given by

$$Z \;=\; \int [d\overline{\psi}\; d\psi] \int [d\phi]\; \mathrm{e}^{-S(\overline{\psi},\psi,\phi)}$$

It is usually believed that

"... there is no way to represent Grassmann variables on a computer so we need to integrate them away! ... "

So in the conventional approach

$$Z = \int [d\phi] \sum_{[\theta]} \;\mathrm{Det}(M([\theta,\phi]))$$

auxiliary field

For Monte Carlo purposes this is a very inefficient approach.

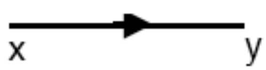
But, recently a new and a more natural approach has emerged!

Grassmann variables are used to generate fermion world line configurations

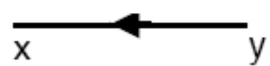
For example consider a single bond term with one flavor

$$e^{\eta_{xy}[\overline{\psi}_x\psi_y e^{i\phi_{xy}} - \overline{\psi}_y\psi_x e^{-i\phi_{xy}}]} = 1 + \eta_{xy}\overline{\psi}_x\psi_y e^{i\phi_{xy}} - \eta_{xy}\overline{\psi}_y\psi_x e^{-i\phi_{xy}} + \overline{\psi}_x\psi_x\overline{\psi}_y\psi_y$$

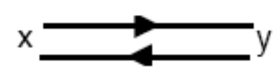
Pictorially the terms with Grassmann variables can be represented as



$$\eta_{xy}\overline{\psi}_x\psi_y e^{i\phi_{xy}}$$



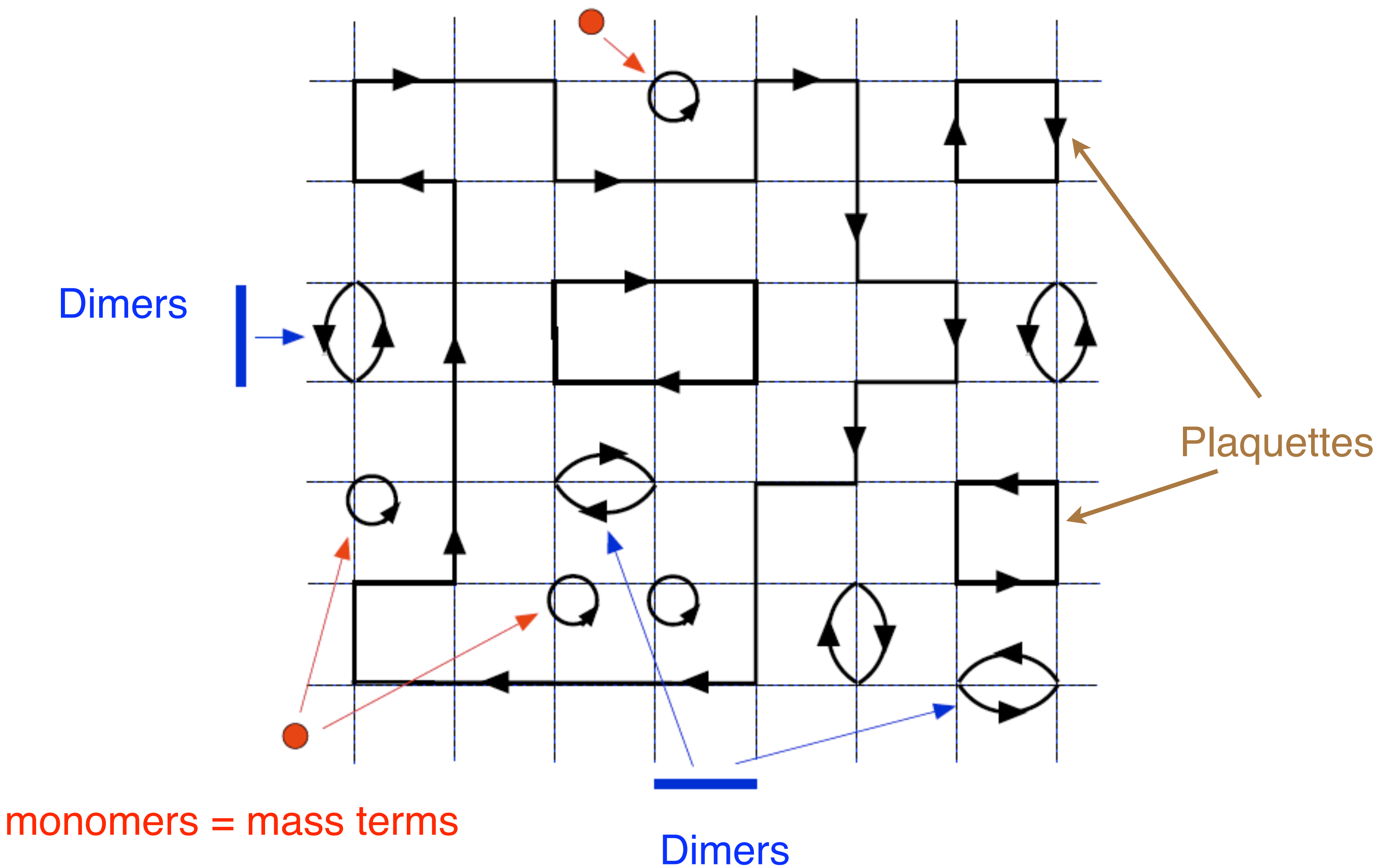
$$-\eta_{yx}\overline{\psi}_y\psi_x e^{-i\phi_{xy}}$$



$$-\overline{\psi}_x\psi_y\overline{\psi}_y\psi_x$$
$$= \psi_x\overline{\psi}_x\psi_y\overline{\psi}_y$$

With this representation can be generalized

Fermion World-line configurations for free staggered fermions



Thus, the fermionic partition function can be written as

$$Z = \sum_{C \in \text{fermion loops}} \text{Sign}([C])\, W([C])$$

The sign function depends on the topology of the loop:

1. There is a negative sign for every backward bond.
2. There is a sign factor that comes from local phases.
3. Every fermion loop is given a negative sign.

This representation is useful in Monte Carlo only if sign problems can be solved

In our model "all" signs cancel and the problem is equivalent to the statistical mechanics of a class monomer-dimer configurations which we call MDPI configurations.

MDPI configurations

MDPI: Monomer Dimer Pion-loop Instanton

Site variables:

Instanton: $I(x) = 0, 1$

u-monomers: $n_u(x) = 0, 1$

d-monomers: $n_d(x) = 0, 1$

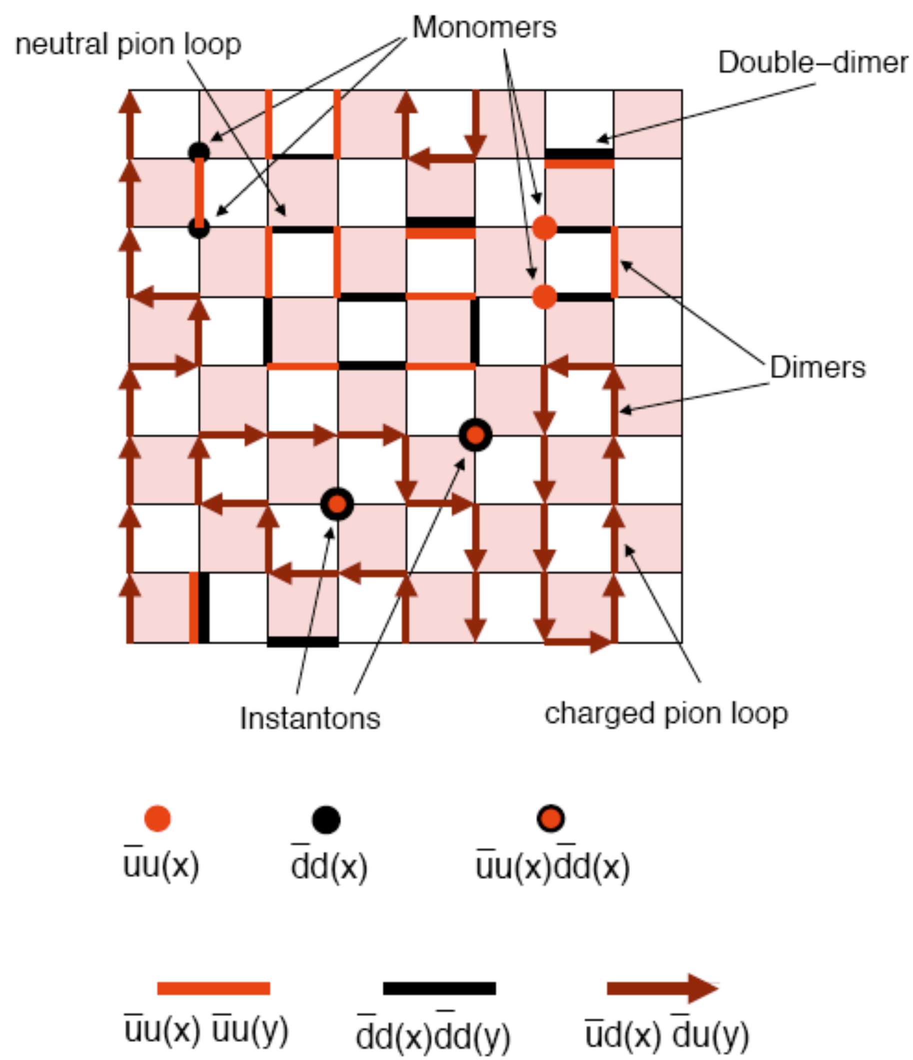
link variables:

uu-dimers $\pi^u_\mu(x) = 0, 1$

dd-dimers $\pi^d_\mu(x) = 0, 1$

ud-dimers $\pi^1_\mu(x) = -1, 0, 1$

There are constraints!



Constraints of MDPI configurations

$$\sum_{\mu} \pi^1_\mu(x) \;=\; 0$$

$$2I(x) + \sum_\mu \left[\pi^u_\mu(x) + \pi^d_\mu(x) + n^u(x) + n^d(x)\right] + \sum_\mu \left|\pi^1_\mu(x)\right| \;=\; 2$$

$$n_u(x) + \sum_\mu \left[\pi^u_\mu(x) - \pi^d_\mu(x)\right] - n_d(x) \;=\; 0$$

MDPI representation of the model

$$Z = \sum_{\mathcal{K}} \prod_x m^{n_u(x)+n_d(x)} c^{I(x)} \; T^{|\pi^u_5(x)+\pi^d_5(x)+\pi^1_5(x)|}$$

$$c \equiv \tilde{c} + m^2$$

Sum is over allowed MDPI configurations

Update Algorithms

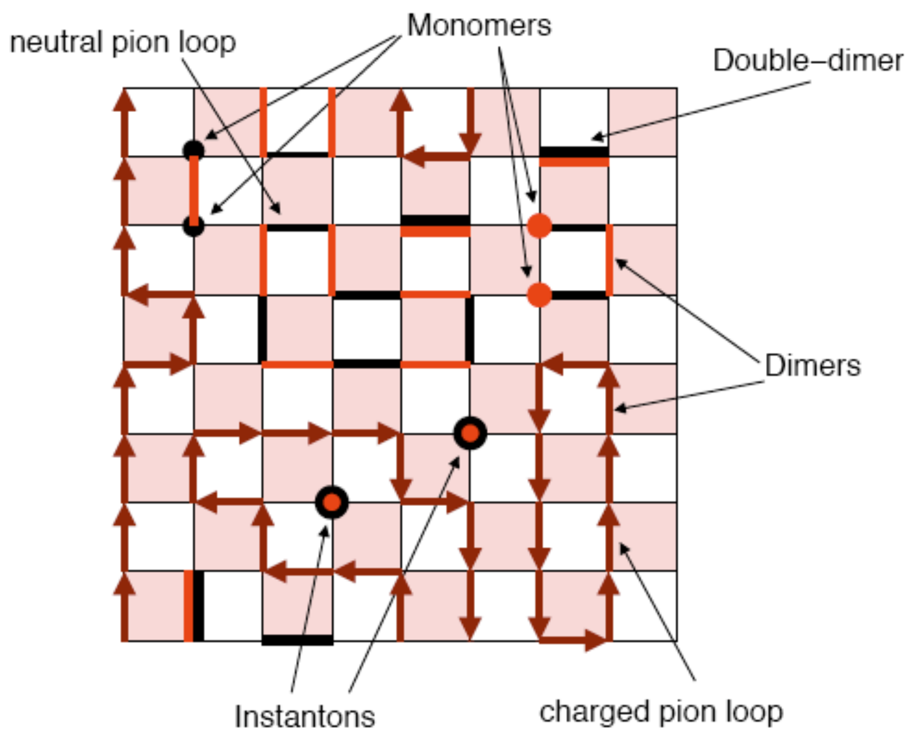
There are four updates in the algorithm

A. Loop Flip update

B. Loop Swap update

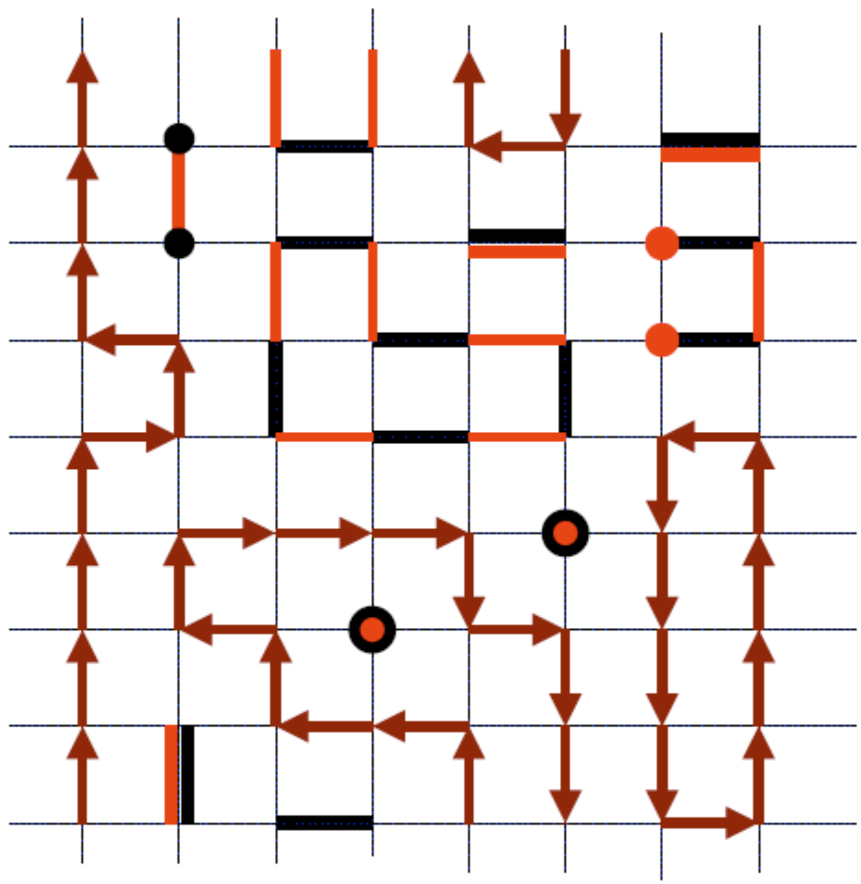
C. Directed Loop update of charged pions

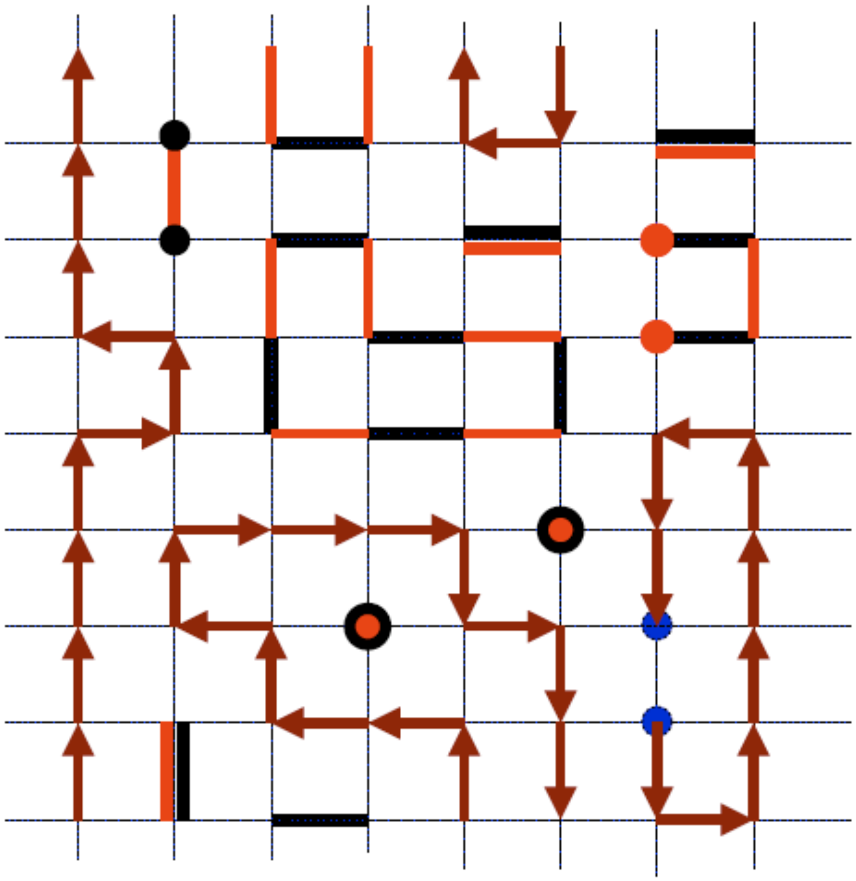
D. Directed Loop update of neutral pions

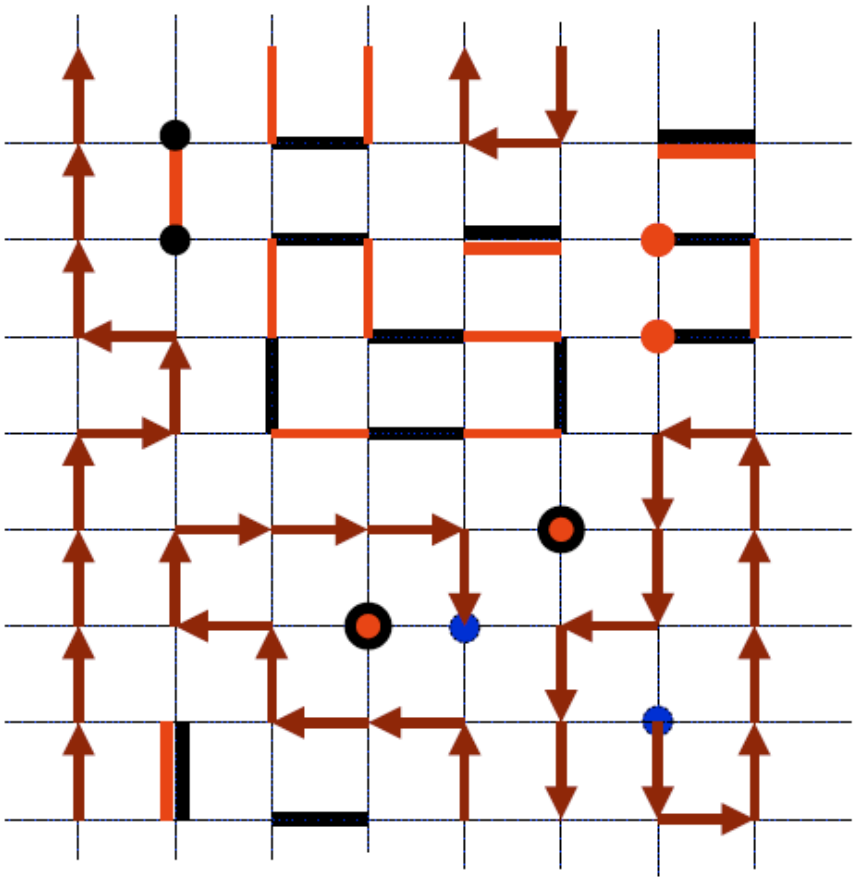


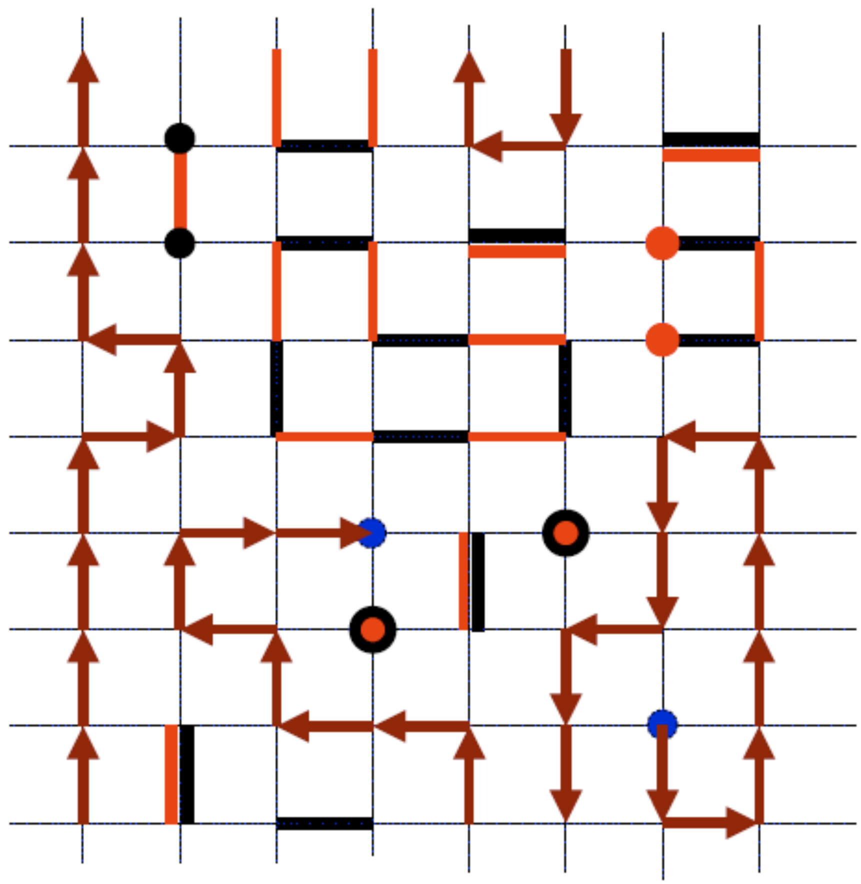
Example of a “Directed Loop update for charged pions”

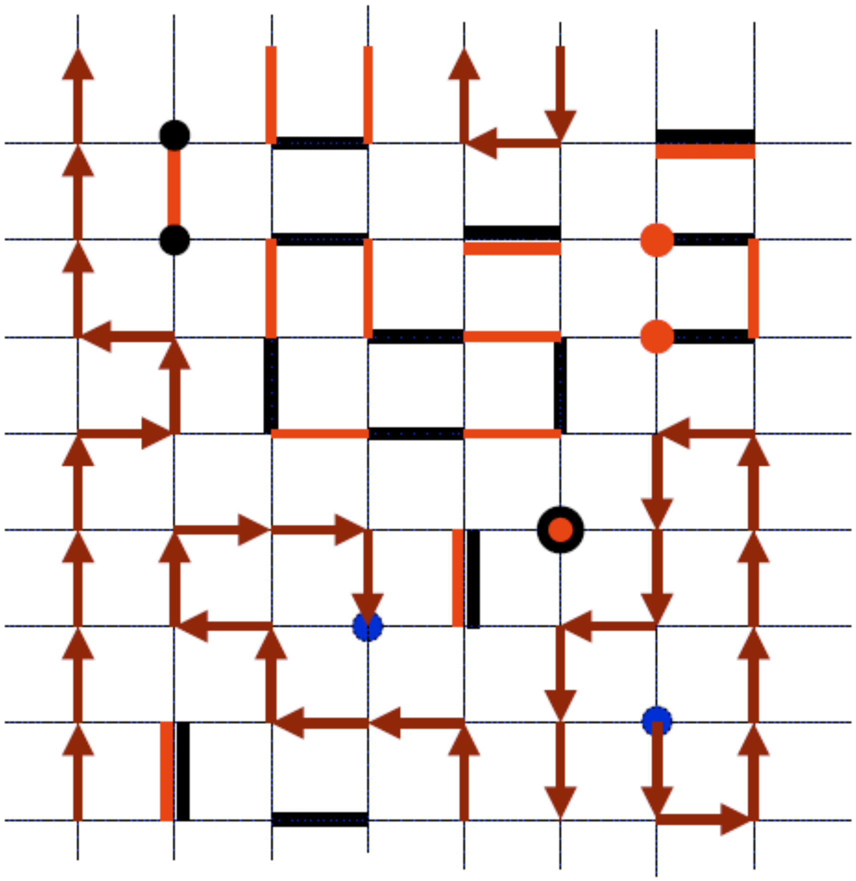
First configuration

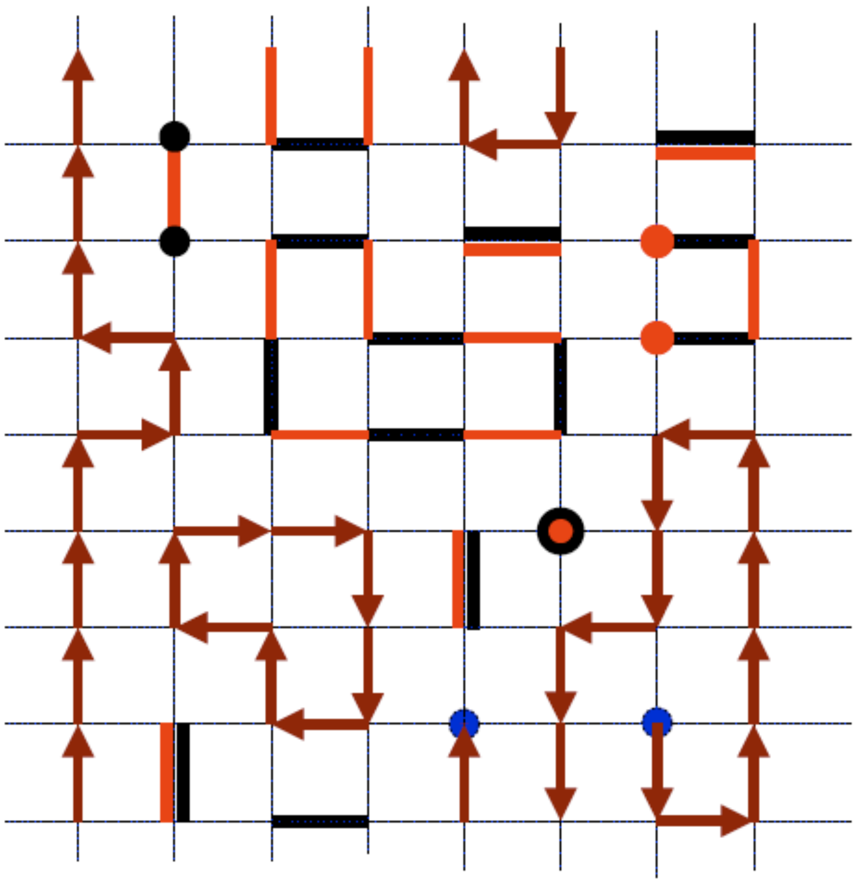


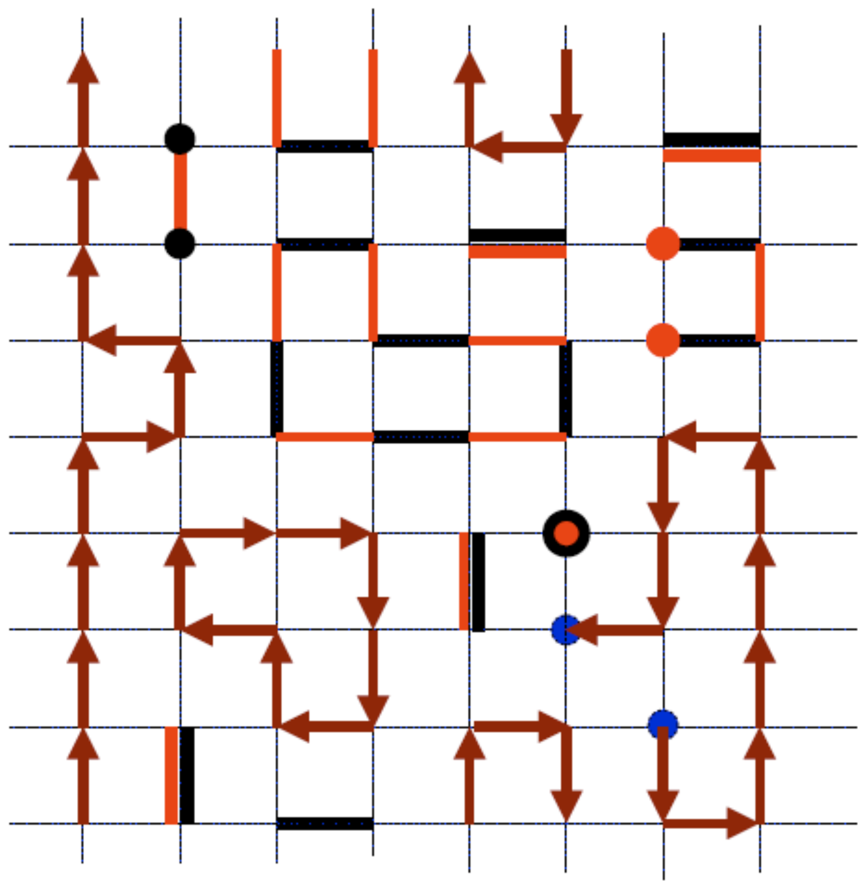


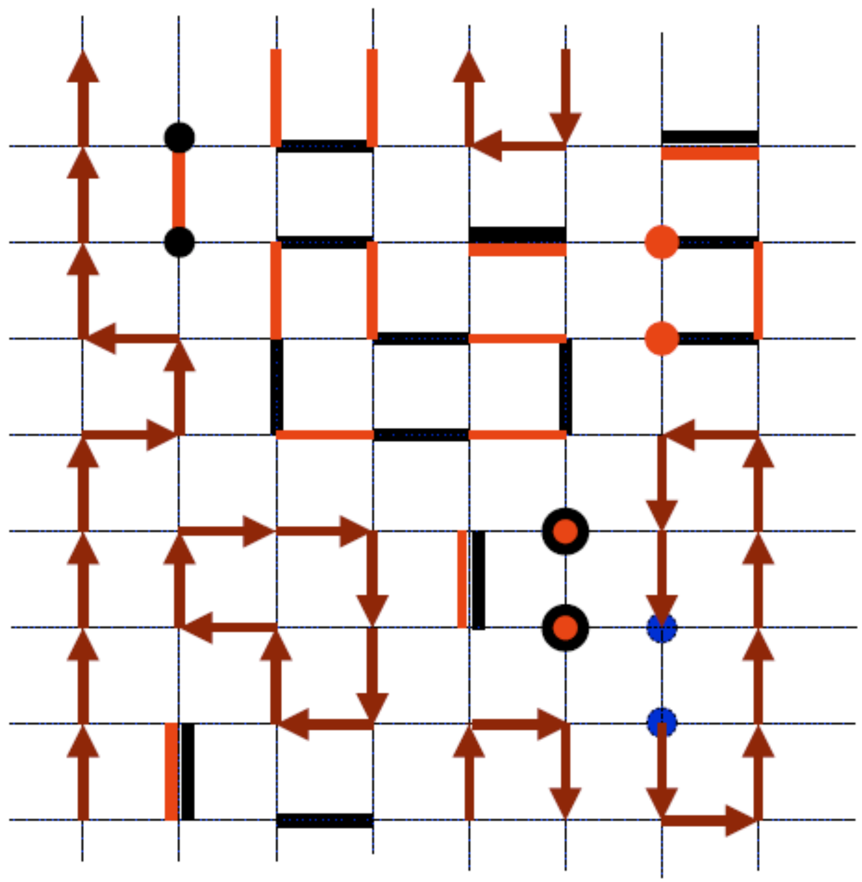




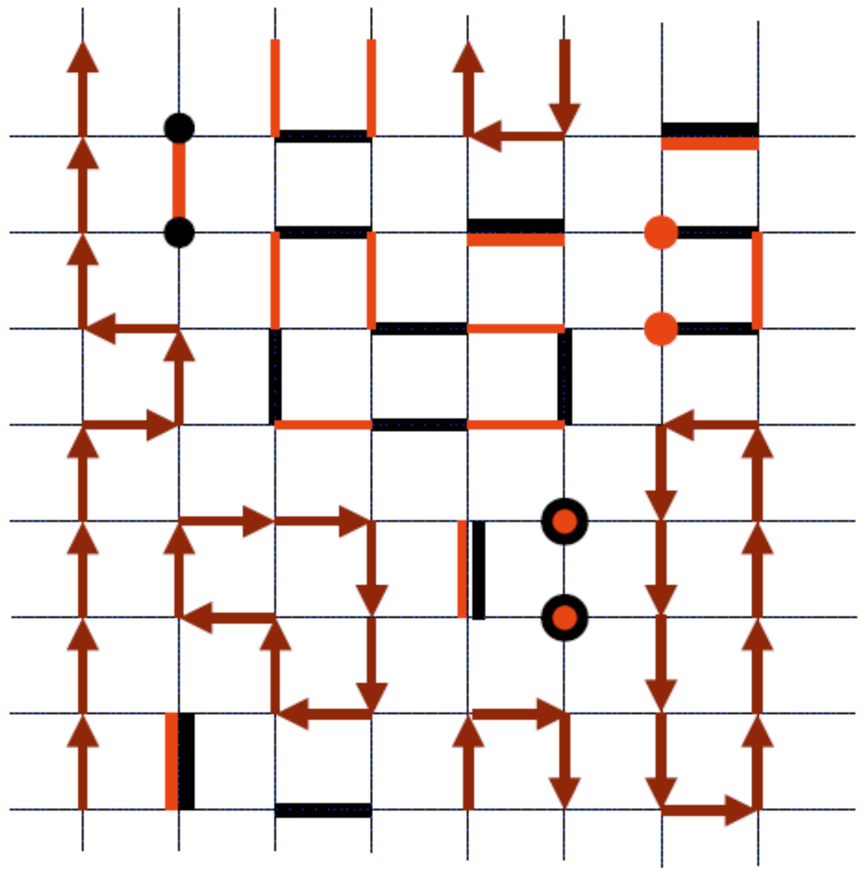








New configuration



Observables

current-current susceptibility

$$Y_i = \frac{1}{dL^d}\left\langle \sum_{\mu=1}^{d}\left(\sum_x J_\mu^i(x)\right)^2\right\rangle$$

continuum notation

Vector Current: $J_\mu^v(x) \longrightarrow Y_v \sim \overline{u}_x\gamma_\mu u_x - \overline{d}_x\gamma_\mu d_x$

Chiral Current: $J_\mu^c(x) \longrightarrow Y_c \sim \overline{u}_x\gamma_\mu\gamma_5 u_x - \overline{d}_x\gamma_\mu\gamma_5 d_x$

Chiral condensate susceptibility: $\chi_\sigma = \frac{1}{L^d}\frac{1}{Z}\frac{\partial^2 Z}{\partial m^2}$

Chiral Perturbation theory predicts
how these quantities behave as a function
of quark mass and lattice size

Chiral perturbation theory

Chiral Lagrangian for two flavor theory at leading order

$$\mathcal{L} = \frac{F^2}{4}\,\mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{m\Sigma}{4}\,\mathrm{Tr}(U + U^\dagger),$$

There are two low energy constants: F and Σ

Our aim is to remain in the broken phase
but at a point close to the phase transition
so that there is a light and narrow sigma
and find the range in the quark mass where
chiral perturbation theory up to one loop is valid
say with five percent errors.

Results

Fixed Running Parameters

$L_5 = 2$

$c = 0.3$ so that the anomalous pion mass is one in lattice units

A careful study of the phase transition reveals that

$T_c = 1.73779(4)$

Hence we choose to fix $T = 1.7$ which is close to T_c
but in the broken phase

We then compute our observables as a
function of L (box size) and m (quark mass)

$\uparrow$

L x L x L x L

ε-regime results

massless quarks: m=0

shape coefficient

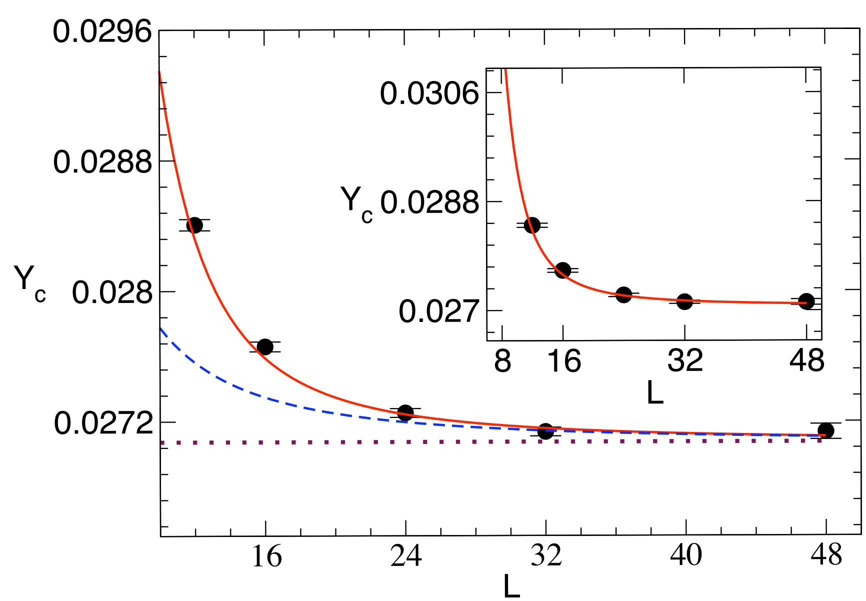
$$Y_c = Y_v = \frac{F^2}{2}\left(1 + \frac{0.14046}{(FL)^2} + \frac{a}{(FL)^4}\right)$$

we can extract F from the fit

$$F = 0.2327(1)$$

$$a = 1.91(9)$$

$$\chi^2/DOF = 1.2$$



ε-regime results

m=0

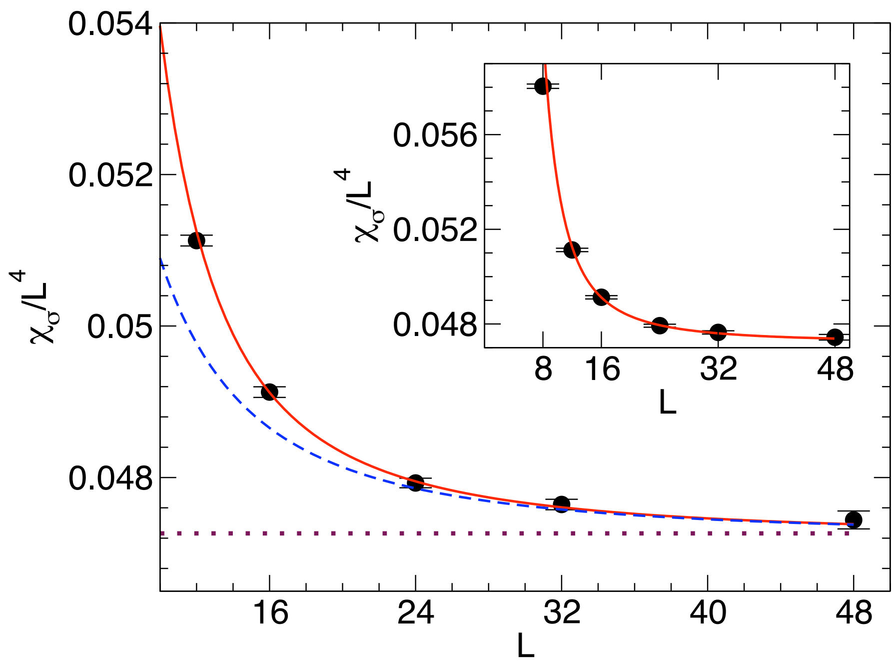
$$\chi_\sigma = \frac{\Sigma^2 L^4}{4}\left(1 + \frac{0.42138}{(FL)^2} + \frac{b}{(FL)^4}\right)$$

we can extract Σ from the fit

$$\Sigma = 0.4346(2)$$

$$b = 1.72(11)$$

$$\chi^2/DOF = 0.2$$



p-regime results

Finite size predictions at a fixed quark mass

$$Y_c = (F_\pi)^2[1 - 2\tilde{g}_1(LM_\pi)\xi + \mathcal{O}(\xi^2)],$$
$$Y_v = (F_\pi)^2\left[-2L\frac{\partial \tilde{g}_1(LM_\pi)}{\partial L}\xi + \mathcal{O}(\xi^2)\right],$$
$$\chi_\sigma = (\langle\bar{q}q\rangle)^2 L^4[1 - 3\tilde{g}_1(LM_\pi)\xi + \mathcal{O}(\xi^2)],$$

$$\xi = M_\pi^2/(16\pi^2 F_\pi^2),$$

$$\tilde{g}_1(\lambda) = \sum_{n_1,n_2,n_3,n_4\neq 0}^{\infty} \frac{4}{\lambda\sqrt{n}} K_1(\lambda\sqrt{n}),$$

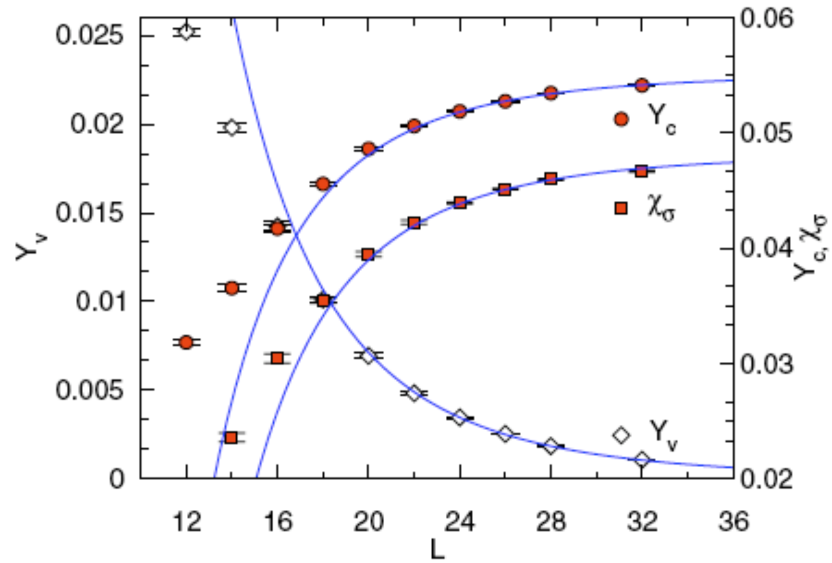
$$n = n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

K_1 is a Bessel function of the second kind

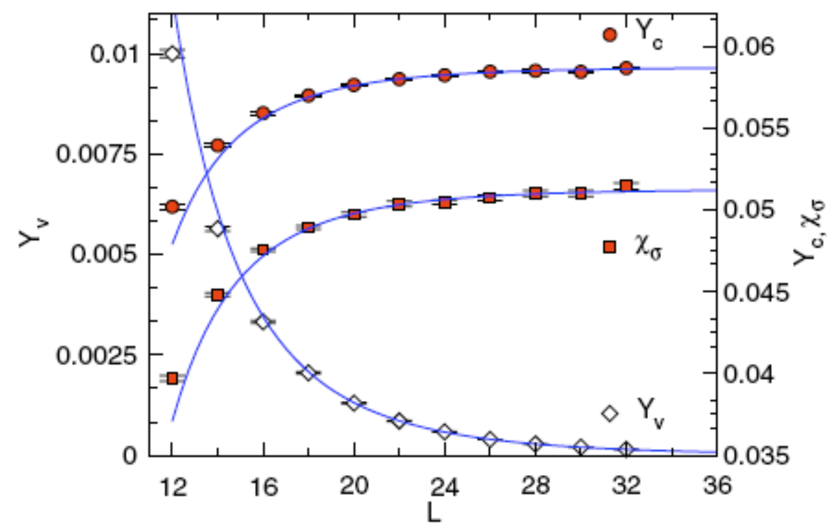
3-parameters to fit: $F_\pi \quad M_\pi \quad \langle\bar{q}q\rangle$

At small masses everything fits beautifully!

m=0.0002



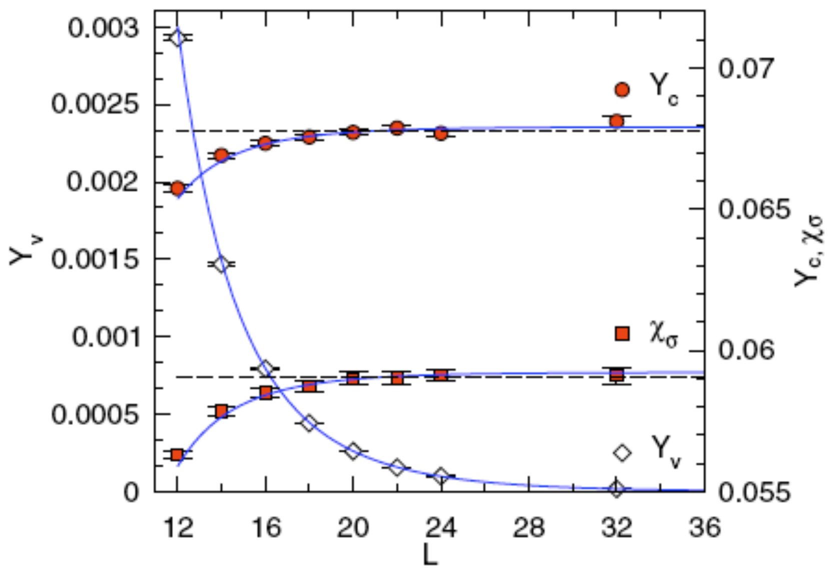
m=0.001



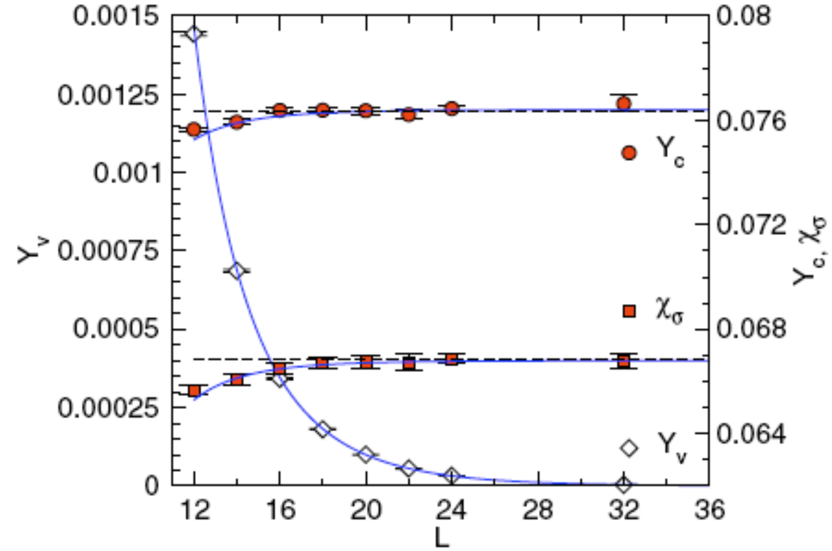
m	$\langle \bar{q}q \rangle$	F_π	M_π	χ^2	Fit range
0.0002	0.4392(2)	0.2348(1)	0.0400(2)	2.5	$24 \le L \le 32$
0.0005	0.4441(2)	0.2377(1)	0.0627(2)	1.1	$24 \le L \le 32$
0.0008	0.4499(2)	0.2406(1)	0.0789(1)	0.9	$22 \le L \le 32$
0.0010	0.4528(2)	0.2423(1)	0.0878(1)	0.8	$18 \le L \le 32$
0.0015	0.4606(2)	0.2467(1)	0.1070(2)	1.3	$18 \le L \le 32$
0.0020	0.4678(2)	0.2501(1)	0.1220(2)	1.8	$20 \le L \le 32$
0.0025	0.4740(2)	0.2538(1)	0.1356(2)	1.6	$16 \le L \le 32$
0.0035	0.4867(2)	0.2606(1)	0.1584(2)	0.9	$16 \le L \le 32$

At slightly larger masses things flatten out quickly

m=0.0035



m=0.0065



m	$\langle \bar{q}q \rangle$	χ^2	F_π	χ^2	M_π	χ^2
0.0020	0.4668(3)	1.2	0.2498(1)	0.1	0.1226(2)	0.6
0.0025	0.4728(3)	0.7	0.2536(2)	0.9	0.1356(2)	1.6
0.0035	0.4861(3)	0.1	0.2603(1)	1.5	0.1584(2)	1.7
0.0050	0.5024(3)	0.2	0.2690(2)	1.1	0.1860(3)	0.7
0.0065	0.5170(3)	0.1	0.2764(2)	0.7	0.2083(4)	0.5
0.0075	0.5247(3)	0.2	0.2807(2)	1.6	0.2219(4)	0.9
0.0100	0.5433(2)	0.7	0.2912(2)	0.1	0.2521(5)	1.8

1-loop chiral peturbation theory

Prediction

$$F_\pi = F[1 - \xi' \log \xi' + 2\xi' c_F],$$
$$\langle \bar{q} q \rangle = \Sigma[1 - \tfrac{3}{2}\xi' \log \xi' + 3\xi' c_\Sigma],$$
$$M_\pi^2 = M^2[1 + \tfrac{1}{2}\xi' \log \xi' - \xi' c_M],$$

$$\xi' = M^2/(16\pi^2 F^2) \qquad\qquad M^2 = m\Sigma/F^2$$

Σ	F	c_Σ	c_F	c_M	χ^2
0.4354(3)	0.2329(2)	11.9(3)	19.3(5)	39(3)	1.1
0.4351(5)	0.2331(4)	12.3(5)	18.9(9)	37(3)	1.6

Matches very nicely with ε-region results

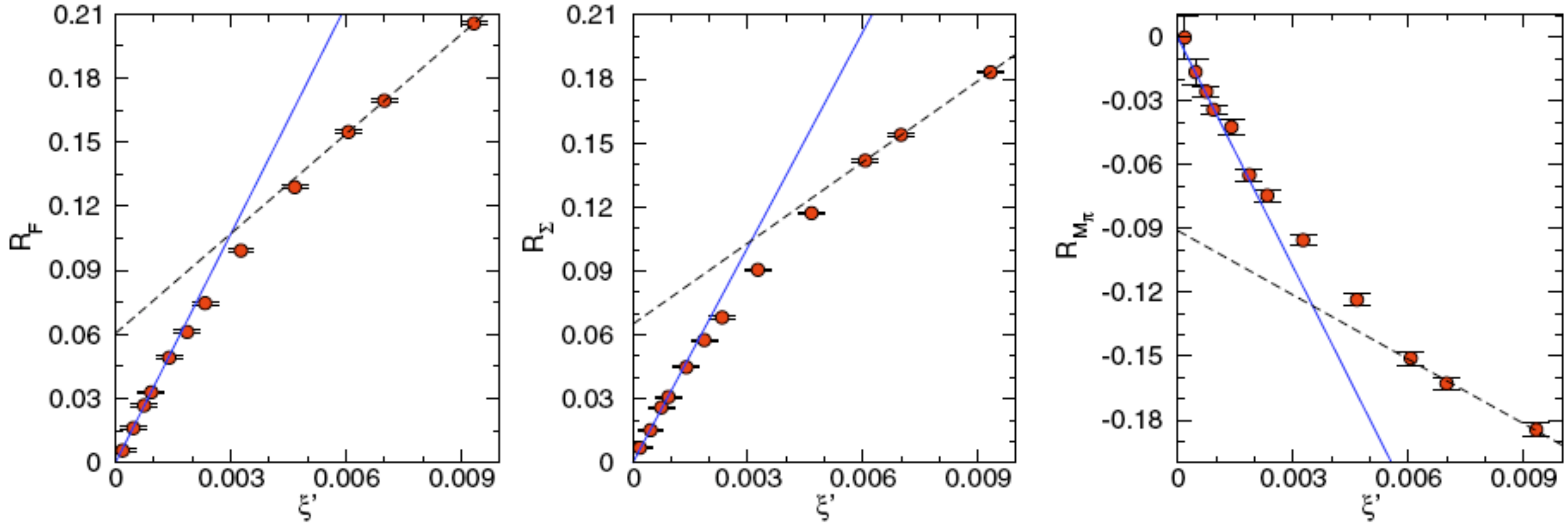
$$\Sigma = 0.4346(2) \qquad F = 0.2327(1)$$

Region of 1-loop chiral perturbaton theory

$$F_\pi = F[1 - \xi' \log \xi' + 2\xi' c_F],$$
$$\langle \bar{q}q \rangle = \Sigma[1 - \tfrac{3}{2}\xi' \log \xi' + 3\xi' c_\Sigma],$$
$$M_\pi^2 = M^2[1 + \tfrac{1}{2}\xi' \log \xi' - \xi' c_M],$$

$$R_F \equiv F_\pi/F - 1 + \xi' \log \xi',$$
$$R_\Sigma \equiv \langle \bar{q}q \rangle/\Sigma - 1 + 3\xi' \log \xi'/2,$$
$$R_M \equiv M_\pi^2/M^2 - 1 - \xi' \log(\xi')/2.$$

R's linearly go to zero
in the region of 1-loop chiral perturbation theory



If 5% or less error is tolerated
ξ'≤0.006 is needed for 1-loop chiral perturbation to be valid!

ξ'≥0.006 another linear region!

A knee is present at ξ' =0.0035.

What is the reason?

σ-resonance and chiral pert. theory

In a weakly coupled linear sigma model one can show

Gockeler et. al., PLB 273, 450 (1991)
Hasenfratz et. al., NPB 356, 332 (1991)

$$c_\Sigma = \log(M_R/4\pi F) - \frac{7}{6} + \frac{8\pi^2}{3g_R},$$

$$c_M = \log(M_R/4\pi F) - \frac{7}{3} + \frac{8\pi^2}{g_R},$$

$$g_R = M_R^2/2F^2$$

where

$$M_\sigma^2 = M_R^2\left[1 + \frac{g_R}{16\pi^2}(3\pi\sqrt{3} - 13)\right].$$

Here M_σ is that mass of the σ particle

As M_σ becomes small
the region of validity of chiral Pert. theory shrinks

Using C_Σ= 12 and C_M=39 we get $M_\sigma/F \sim 2$

At the knee (i.e., m=0.0035) we find $M_\pi/F \sim 0.6$

Thus the knee occurs roughly when $M_\pi = M_\sigma/3$

In the current model
1-loop Chiral perturbation theory
begins to break down roughly when
$M_\pi \geq M_\sigma/3$

Take-home message

All resonances do indeed play an important role
in determining the window of Ch.P.T
Here we showed a rather indirect evidence.
Future: A more direct study.

Important to match low energy constants
from the ε-regime and the p-regime.

Non-perturbative lattice calculations in models
with the same symmetries and
the same low energy constants
may be useful to understand resonance physics.