Evidence of phase transition in dense matter from QCD Sum Rules

S. Mallik and Sourav Sarkar

January 8, 2009

Introduction

QCD predicts strong interaction among quarks and gluons at relatively large distance, while they become asymptotically free at short distance.
This feature leads to the common belief that with the rise of temperature and/or density, nuclear matter undergoes one or more phase transitions, restoring the spontaneously broken chiral symmetry and liberating colour to form the quark-gluon plasma.

While the corresponding order parameters, namely the quark condensate and the Polyakov loop, can in principle be calculated with sufficient accuracy at finite temperature, such a calculation at finite chemical potential is problematic.

Here we describe an analytic method to investigate the change in nuclear matter as its density rises.
We find the density dependence of the coupling of an external source, namely the nucleon current $\eta(x)$ with nucleons in the medium.
This current has the quantum numbers of the nucleon and is composed of three quark fields, which we shall spell out later
B.L. Ioffe, Nucl. Phys. B188, 317 (1981);
Y. Chung et al, Nucl. Phys. B197, 55 (1982).

In exact terms, we consider the ensemble average of the two-point function of the nucleon current at finite chemical potential μ and temperature β^{-1},

$$\Pi(E,\vec{p}) = i\int dtd^3x\, e^{i(Et-\vec{p}\cdot\vec{x})}\langle T\eta(x)\bar{\eta}(0)\rangle \tag{1}$$

where for any operator O

$$\langle O\rangle = Tr[e^{-\beta(H-\mu N)}O]/\mathcal{Z}, \qquad \mathcal{Z} = Tr[e^{-\beta(H-\mu N)}]$$

The coupling parameter stated above appears as the residue of the two-point function at the nucleon pole.

This idea is similar to that of Leutwyler and Smilga
Nucl. Phys. B342, 302 (1990), who found the temperature dependence of the residue at the nucleon pole by evaluating the one-loop diagrams in the framework of the (chiral) effective field theory. Their calculation requires only the πN interaction.

The difficulty at finite density is that in addition to πN interaction, it involves NN interaction also.

But whereas the effective theory for the πN interaction is satisfactory even to one loop, the same for the NN interaction is problematic, even when higher loops are included
D.H. Politzer et al, Nucl. Phys. B375, 507 (1992);
S. Weinberg, Nucl. Phys. B363, 3 (1991).

To overcome this difficulty, we we write down the QCD sum rules for the amplitudes.
Roughly speaking, these are obtained by evaluating the amplitudes in two ways:
One is to write Lehmann representations for them in terms of the physical (hadronic) spectrum of the system.
The other is to evaluate the operator product expansion at large momenta.
The two representations are then equated in a suitable region.
In this way one may get a sum rule for the density dependent coupling parameter.

It still involves the NN interaction in the form of self-energies of the quasi-particles.
But These quantities have already been calculated by variational and Brueckner type approaches using the phenomenological NN interaction potentials.
R. Brockmann and Machleidt, Phys. Rev. C 42, 1965 (1990);
de tar Haar and Malfliet, Phys. Rep.
We take over these results in our calculation.

To increase the sensitivity of the sum rule, we do not evaluate just the sum rule in medium, but subtract from it the corresponding vacuum sum rule.
Then the non-leading contributions in the low energy region are expected to be important.
Also the sum rules will involve only the medium dependent quantities.

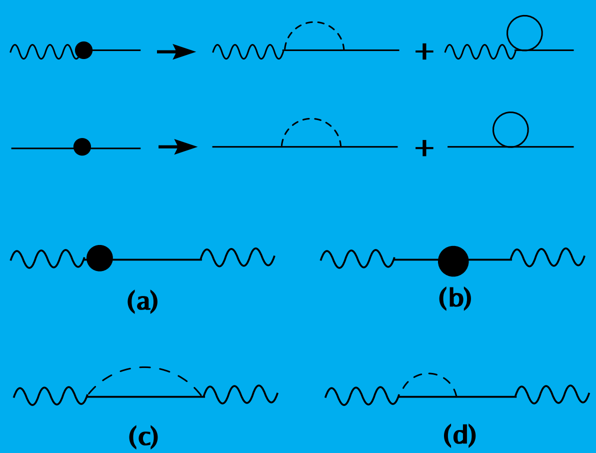
Now a technical remark. We shall use the real time version of the field theory in a medium, where a two-point function assumes the form of a 2×2 matrix.
But the dynamics is given essentially by a single analytic function.
Thus if $\Pi_{11}(p)$ is the 11-component of the matrix amplitude, the corresponding analytic function $\Pi(E, \vec{p})$ has the spectral representation

$$\Pi(E, \vec{p}) = \frac{1}{\pi} \int dE' \frac{\coth\{\beta(E' - \mu)/2\} \, \mathrm{Im}\Pi_{11}(E', \vec{p})}{E' - E - iE'\epsilon}. \tag{2}$$

N.P. Landsmann and Ch. G. van Weert, Phys. Rep. 145, 141 (1987)

Sum Rule

Let us begin with the spectral side of the sum rules. We construct it from the contribution of the Feynman diagrams of Figs. 1 and 2, in which we draw a blob where the NN interaction is involved.



The free propagator of nucleon of Fig. 1(a) contributes

$$-\frac{\lambda^2}{\not{p} - m + i\epsilon}$$

Here the coupling λ is given by the matrix element

$$\langle 0|\eta(x)|N(p)\rangle = \lambda u(p) e^{ip\cdot x}$$

where $u(p)$ is the positive energy Dirac spinor. Figs.1(b) and (c) and (d) bring in the self energy and the vertex correction, modifying the free propagation to

$$-\frac{\lambda^{*2}}{\not{p} - m - \Sigma(p)} \qquad (3)$$

where λ^* is is the corrected coupling parameter and $\Sigma(p)$ is the self-energy matrix.

We restrict to $\vec{p} = 0$ and decompose Σ into its scalar and vector parts as $\Sigma = \Sigma_S + \gamma^0 \Sigma_V$. As already stated, Σ_S and Σ_V have been calculated by nuclear physicists.
Then Eq. (3) becomes

$$-\lambda^{*2} \frac{\gamma^0(E - \Sigma_V) + m^*}{(E - m_1)(E - m_2)} \qquad (4)$$

where $m^* = m + \Sigma_S$. The quasi-particle poles at $m_1 = \Sigma_V + m^*$ and $m_2 = \Sigma_V - m^*$ correspond to the nucleon and the antinucleon.

Next we calculate the loop diagram of Fig. 1(e). We have the general expression for the imaginary part,

$$\begin{aligned}\mathrm{Im}\Pi_{11}^{(e)} &= -\pi\tanh\{\beta(E-\mu)/2\}\left(\frac{3\lambda^2}{4F_\pi^2}\right)\int\frac{d^3q}{(2\pi)^2 4\omega_1\omega_2}\\ &\Big[(-\gamma^0\omega_1+m)\{(1-n_+ + n)\delta(E-\omega_1-\omega_2)+(n_+ + n)\delta(E-\omega_1+\omega_2)\}\\ &+(\gamma^0\omega_1+m)\{(-1+n_- - n)\delta(E+\omega_1+\omega_2)-(n_- + n)\delta(E+\omega_1-\omega_2)\}\Big] \end{aligned} \quad (5)$$

where $\omega_1=\sqrt{m^2+\vec{q}^2}$, $\omega_2=\sqrt{(m_\pi^2+\vec{q}^2)}$ and $n_\pm$ and n are respectively the distribution functions for nucleons, antinucleons and pions,

$$n_\pm(\omega_1)=\frac{1}{e^{\beta(\omega_1\mp\mu)}+1}\,,\qquad n(\omega_2)=\frac{1}{e^{\beta\omega}-1} \quad (6)$$

Terms without the n's are the vacuum contributions, which we reject, as already stated above.

Further, we restrict to zero temperature so that we have to calculate only the terms proportional to $n_+\to\theta(\mu-\omega_1)$.

We thus get

$$\mathrm{Im}\Pi_{11}^{(e)} = \pi \tanh\{\beta(E-\mu)\}/2f(E)(-1,1) \tag{7}$$

where

$$f(E) = \frac{3\lambda^2}{32\pi^2 F_\pi^2 E}\sqrt{\omega^2 - m^2}(\gamma^0\omega - m)$$

and $(-1,1)$ denotes the sign of the imaginary part on the Landau cut, $\mu - \sqrt{\mu^2 - m^2 + m_\pi^2} \le E \le m - m_\pi$ and
the unitarity cut, $m + m_\pi \le E \le \mu + \sqrt{\mu^2 - m^2 + m_\pi^2}$ respectively.
Then as already stated, the desired analytic function is given by

$$\Pi^{(e)}(E) = \int_C \frac{dE'\, f(E')}{E' - E - i\eta\epsilon(E')} \tag{8}$$

where the subscript C denotes the difference of two integrals,

$$\int_C = \int_{Landau} - \int_{Unitary}$$

Similarly the Figs. 1(f,g) give

$$\Pi^{(f+g)}(E) = 2\int_c \frac{dE'\, g(E')}{E' - E - i\eta\epsilon(E')} \cdot \frac{\gamma^0 E + m}{E^2 - m^2 - i\epsilon} \tag{9}$$

where

$$g(E) = \frac{3\lambda^2 g_A}{32\pi^2 F_\pi^2 E}\sqrt{\omega^2 - m^2}\{(m^2 - E\omega) + \gamma^0 m(E - \omega)\}$$

Of course, we have to subtract out the nucleon pole from this contribution, as the complete nucleon pole contribution is represented by Eq. (4).

B At this point, we follow a suggestion by Cohen et al [Phys. Rev. C49, 464 (1994). Of the two quasi particle poles in Eq. (4), the one corresponding to antinucleon is not well determined as it should have a large width due to annihilation processes.
To suppress its contribution they work with a different amplitude which we adopt here. Thus separating the Dirac matrix structure,

$$\Pi(E) = \Pi_1(E) + \gamma^0 \Pi_2(E) \tag{10}$$

we split each of $\Pi_{1,2}$ into even and odd parts,

$$\Pi_i(E) = \Pi_i^{(E)}(E^2) + E\Pi_i^{(O)}(E^2), \qquad i = 1, 2 \tag{11}$$

and deal with the combination,

$$\Pi_i^{(E)}(E^2) - m_2 \Pi_i^{(O)}(E^2) \tag{12}$$

(Recal that m_2 is the pole position of the quasi anti-nucleon.) For the pole term of Eq. (4) this combination gives

$$-\frac{\lambda^{*2} m^*}{E^2 - m_1^2}(1 + \gamma^0) \tag{13}$$

The factor $(1 + \gamma^0)$ shows that it is the positive energy part of the nucleon pole. We reject the other combination $(1 - \gamma^0)$ that is not accurately determined.
We can now collect all the pieces to complete writing the spectral side of the sum rule.

We now turn to the operator side.

In obtaining the operator product expansion, we need the explicit form of the nucleon current $\eta(x)_{D,i}$ with spin and isospin indices D and i. Of the two independent possibilities involving three quark fields, we take here the preferred one, which for proton $(i=1)$ is

$$\eta(x)_{D,i} = \epsilon^{abc}(u^{aT}(x)C\gamma^\mu u^b(x))(\gamma_5\gamma_\mu d^c(x))_D$$

where C is the charge conjugation matrix and a, b, c are the colour indices. The contributing operators of lowest dimension in the expansion of the operator product of nucleon currents are $\bar{q}$, $\bar{q}\not{u}q$ (=$q^\dagger q$ in the rest frame of matter, $u^\mu = (1,0,0,0)$). Then there are the operators of dimension four, namely $(\alpha_s/\pi)G^a_{\mu\nu}G^{\mu\nu a}$, $\Theta^{f,g} \equiv u^\mu u^\nu \Theta^{f,g}_{\mu\nu}$ with

$$\Theta^f_{\mu\nu} = i\bar{q}\gamma_\mu D_\nu q - \frac{\hat{m}}{4}g_{\mu\nu}\bar{q}q \qquad (14)$$

$$\Theta^g_{\mu\nu} = -G^c_{\mu\lambda}G^{\lambda c}_\nu + \frac{1}{4}g_{\mu\nu}G^c_{\alpha\beta}G^{\alpha\beta c} \qquad (15)$$

where $\hat{m}$ is the average quark mass of u and d quarks and $G^a_{\mu\nu}$ are the gluon field strengths. Of these we retain only the operator Θ^f, as the coefficients of the remaining are small, arising from three contractions of the quark operators in the two point functions.

Among dimension five operators, the coefficient of $\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q$ turns out to be zero; the other operators bringing in small contributions to the sum rules and are ignored.

Of dimension six operators, we retain only the quark operators that have no derivatives or gluon fields.
In the case of vacuum sum rules, their vacuum expectation values may be related to the square of the vacuum expectation value of $\bar{q}q$, using factorisation or vacuum saturation. A similar approximation can be made for the ensemble average but simple model estimates seem to suggest that it may not be as good as the vacuum case.

At this point we follow Cohen et al to use factorisation and then replace the scalar-scalar condensate by the parametrisation,

$$\langle\bar{q}q\rangle^2 \rightarrow (1-f)\langle 0|\bar{q}q|0\rangle^2 + f\langle\bar{q}q\rangle^2 \tag{16}$$

where f is a real parameter.

In terms of the above operators, the operator expansion gives

$$\Pi(E,\vec{0}) \rightarrow -\frac{2E}{3E^2}(\gamma^0 s^2 + 2sv) + \frac{1}{4\pi^2}(s + 4v\gamma^0)E^2 \ln(-E^2) - \frac{5}{6\pi^2}\gamma^0\langle\Theta^f\rangle E \ln(-E^2) \quad (17)$$

where to first order in nuclear density,

$$s \equiv \langle \bar{u}u\rangle = \langle \bar{d}d\rangle = \langle 0|\bar{q}q|0\rangle + \frac{\sigma}{2\hat{m}}\overline{n}\,; \qquad v \equiv \langle u^\dagger u\rangle = \frac{3}{2}\overline{n}\,; \qquad \langle\Theta^f\rangle = \frac{3}{4}mA^f\overline{n}, \quad A^f = 0.62 \quad (18)$$

Here $\overline{n}$ is the nucleon density, related to the Fermi momentum p_F by $\overline{n} = 2p_F^3/(3\pi^2)$. The equilibrium nuclear matter density is $\overline{n} = (110\, MeV)^3$ corresponding to $p_F = 1.36\, fm^{-1}$ The quantity A^f arises in the nucleon matrix element of the quark energy-momentum tensor,

$$\langle p|\Theta^f_{\mu\nu}|p\rangle = 2A^f(p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}p^2)$$

The constant is given by an integral over the nucleon structure function in the deep inelastic region. At $Q^2 = 1\, GeV^2$, the integral gives $A^f = .62$.

A.D. Martin et al, Eur. Phys. J. C4, 463 (1998).

Excluding vacuum contributions and working with the combination of amplitudes given by Eq.(12), we get at Euclidean momenta $E^2 = -Q^2$, $Q^2 > 0$,

$$\Pi^{(E)} - m_2 \Pi^{(O)} \rightarrow -\frac{2m^2}{3Q^2}(\gamma^0 s^2 + 2sv) - \frac{1}{4\pi^2}(s + 4v\gamma^0)Q^2 \ln(Q^2) + \frac{5}{6\pi^2}\gamma^0 \langle \Theta^f \rangle E \ln(Q^2) \quad (19)$$

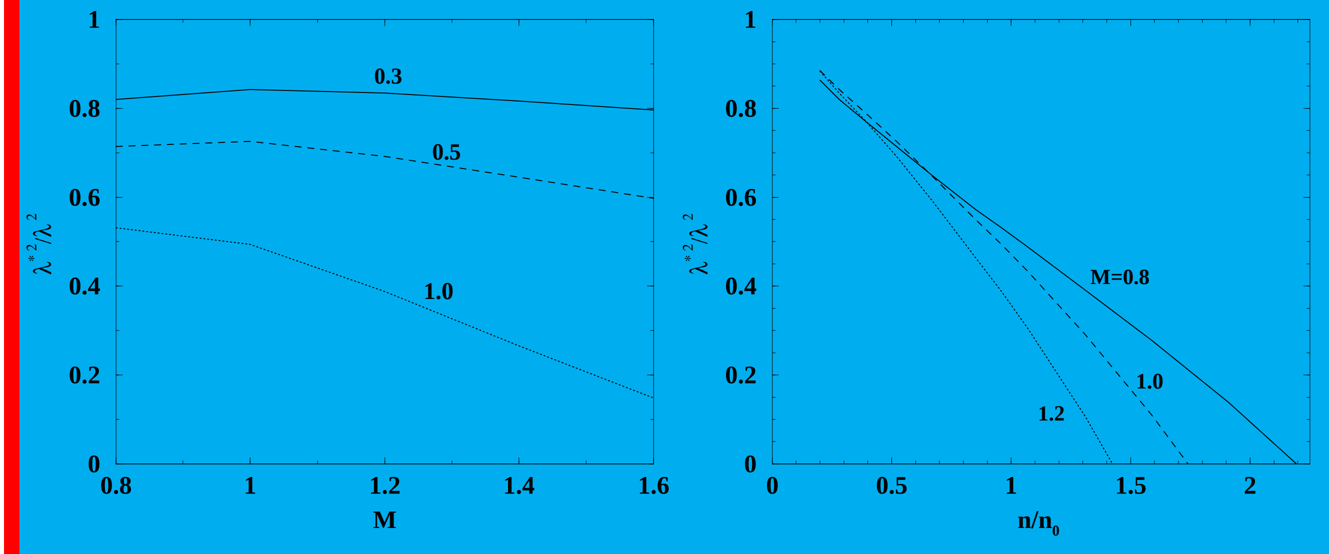
It is now simple to take the Borel transform of the spectral and operator sides and get the desired sum rule

$$\begin{aligned}\lambda^{*2} = \quad & \lambda^2 e^{m_1 2/M^2} \Bigg[\frac{m - m_2}{2m^*} e^{-m^2/M^2} \\ & - \frac{3}{64\pi^2 F_\pi^2 m^*} \int_C \frac{dE}{E}(E - m_2)\sqrt{\omega^2 - m^2}(\omega - m)e^{-E^2/M^2} \\ & + \frac{3g_A}{32\pi^2 F_\pi^2 m^*} \int_C \frac{dE}{E}(E - m_2)\sqrt{\omega^2 - m^2}(\omega - m)^2 \frac{(E + m)}{(E - m)} e^{-E^2/M^2} \\ & - \frac{M^2}{2\lambda^2 m^*} \left\{ \frac{2\langle 0|\bar{u}u|0\rangle m_2}{M^2} \left(1 + \frac{\sigma}{3\hat{m}} f\right) + \frac{M^2}{8\pi^2}\left(\frac{\sigma}{\hat{m}} + 12\right)\right) V_2 + \frac{5mA^f}{8\pi^2} m_2 V_1 \right\} \Bigg] \quad (20)\end{aligned}$$

where $V_1 = 1 - e^{W^2/M^2}$, and $V_2 = 1 - (1 + W^2/M^2)e^{-W^2/M^2}$ The deviations of $V_{1,2}$ from unity represents the contribution from the high energy region on the spectral side. W is a parameter determining the onset of this contribution.

Evaluation of Sum Rule

A preliminary result is shown in Fig.2.



It is seen that λ^* decreases with density. Clearly it cannot be trusted up to the point where it vanishes. Nevertheless, its vanishing at $\overline{n} \simeq 2\overline{n}_0$ does indicate that a phase transition takes place at not too high a density.

Discussion

Let us recall some of the salient features that has gone into the derivation of this sum rule

1. The most important feature of our sum rule is that it not just the sum rule in the medium, but is one obtained from it after subtracting out the corresponding vacuum sum rule. Thus we subtract out the vacuum nucleon pole from the expression (4) and omit vacuum contributions from imaginary parts of loop diagrams like that given by Eq. (5). Also on the operator side, we omit vacuum contributions, in particular, the dominant vacuum contribution corresponding to the unit operator is subtracted out.

2. We also include the density dependent contributions from all one loop diagrams. THe values of nucleon self-energies are not determined by the sum rules, but are taken from earlier calculations by nuclear physicists.

3. We write the sum rule for the amplitude (12) to suppress the contribution from the quasi-antinucleon pole and consider only the sum rules proportional to $(1+\gamma_0)$.

Our work shows that this sum rule, if evaluated correctly, can give significant information regarding the approach of nuclear matter to the quark-gluon phase with the rise in its density.